THE TOP10 ALGORITHMS FROM THE 20TH CENTURY

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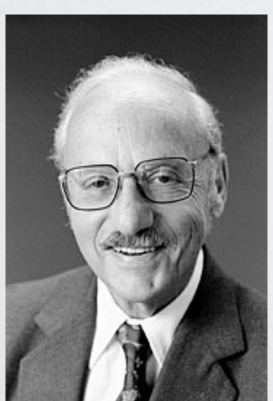
THE TOP 10 LIST

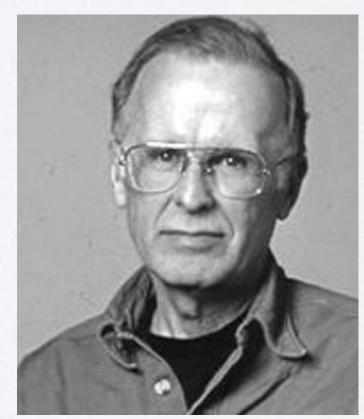
1946: The Metropolis Algorithm
1947: Simplex Method
1950: Krylov Subspace Method
1951: The Decompositional Approach to Matrix Computations
1957: The Fortran Optimizing Compiler
1959: QR Algorithm
1962: Quicksort
1965: Fast Fourier Transform
1977: Integer Relation Detection
1987: Fast Multipole Method

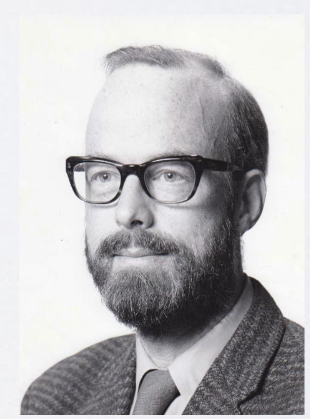

Dantzig


von Neumann


Hestenes


Householder


Backus


Hoare


Greengard

WHAT IS AN ALGORITHM?

Definition:
"An algorithm is a sequence of finite computational steps that transforms an input into an output" [Cormen and Leiserson, 2009]

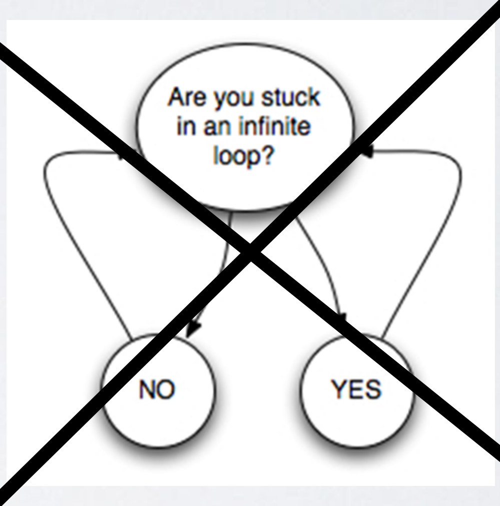
Making tea

Set of instructions

Baking a cake

Recipe

Finite



`while(1), end`

NUMERICAL ANALYSIS

A definition

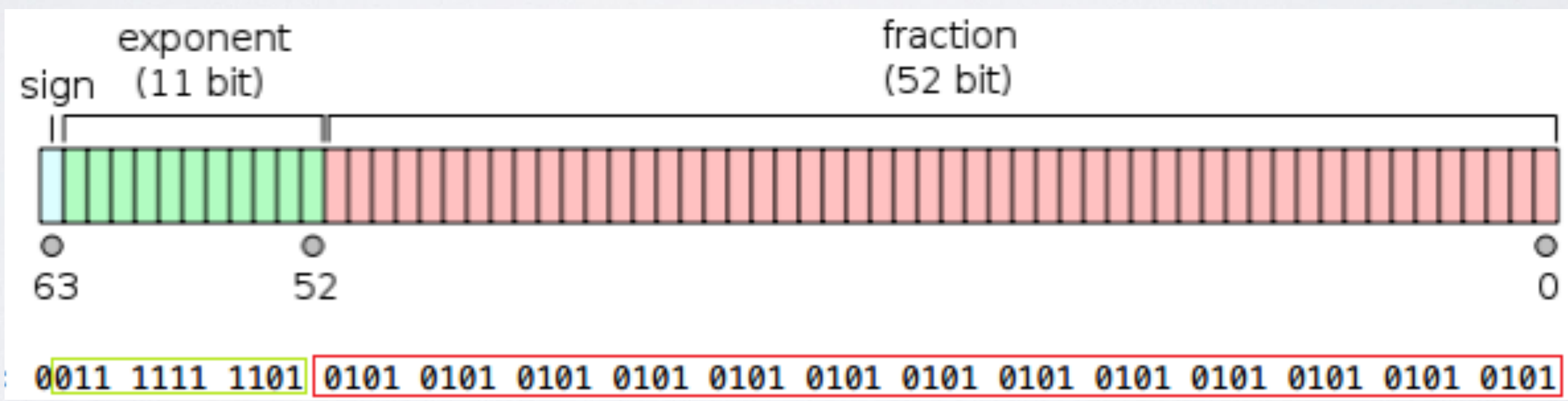
"The study and development of algorithms that use numerical approximation"

How many of the top 10 algorithms are in numerical analysis?

Potentially all of them

Floating point arithmetic

$1/3 \approx$



$$1/3 \approx (-1)^s \left(1 + \sum_{i=1}^{52} b_{52-i} 2^{-i}\right) \times 2^{e-1023}$$

Algorithms implemented in floating point arithmetic are studied and developed by numerical analysts

OVERVIEW OF TALK

A top 10 algorithm

How it works?

How do I use it?

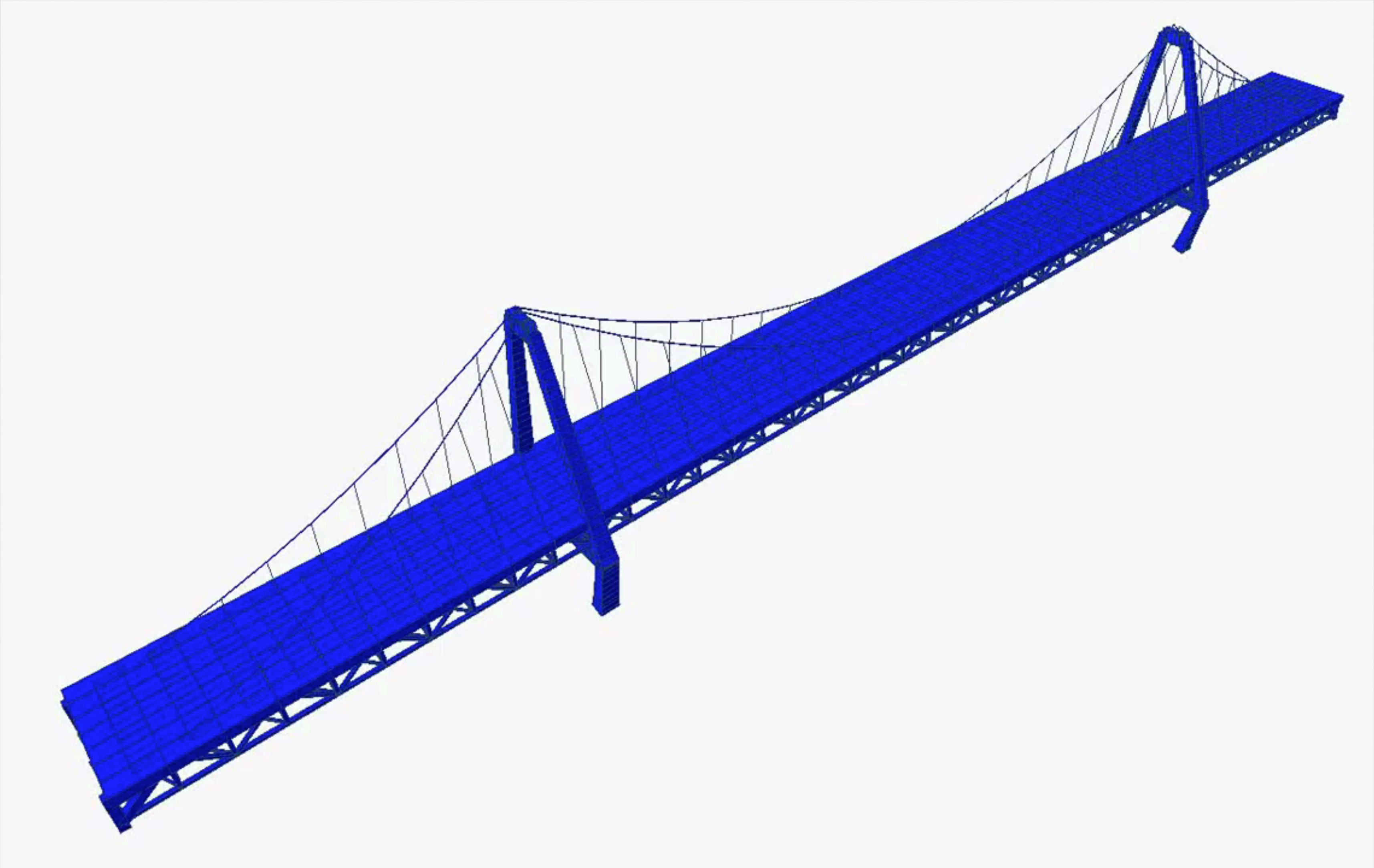
Open problem

1959: QR ALGORITHM

The Tacoma Narrows bridge in Nov 1940

Collapsed in 80km/h winds

NUMERICAL SIMULATIONS

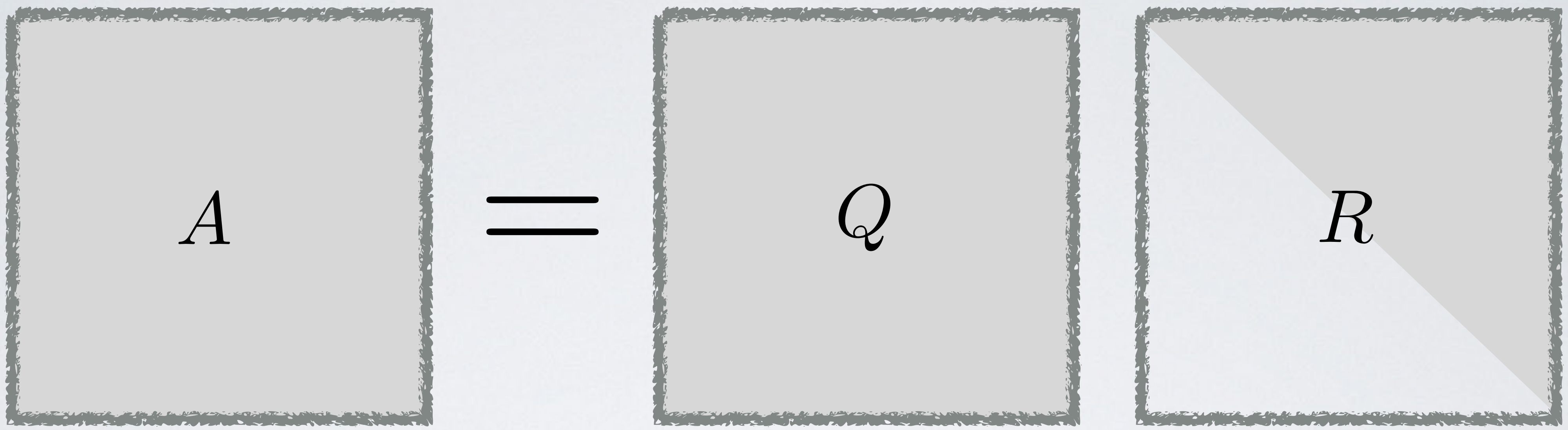


Eigenvalue

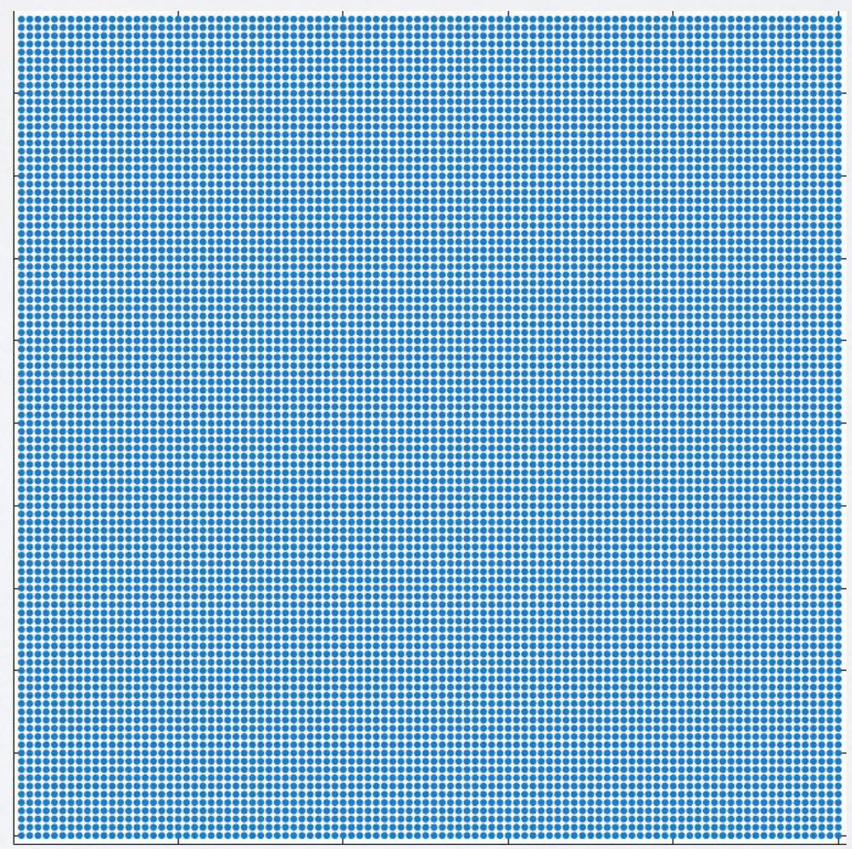
Resonant frequencies are eigenvalues: $A\underline{v} = x\underline{v} \quad \underline{v} \neq 0$

Eigenvector

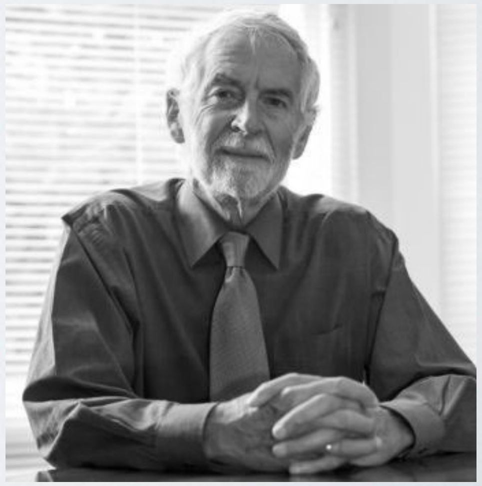
HOW DOES IT WORK?




```
A = symmetric
for k = 1,2,...
   A = Q*R
   A = R*Q
end
```



The final diagonal matrix contains all the eigenvalues



Francis

HOW DO I USE IT?

Rootfinding and global optimization

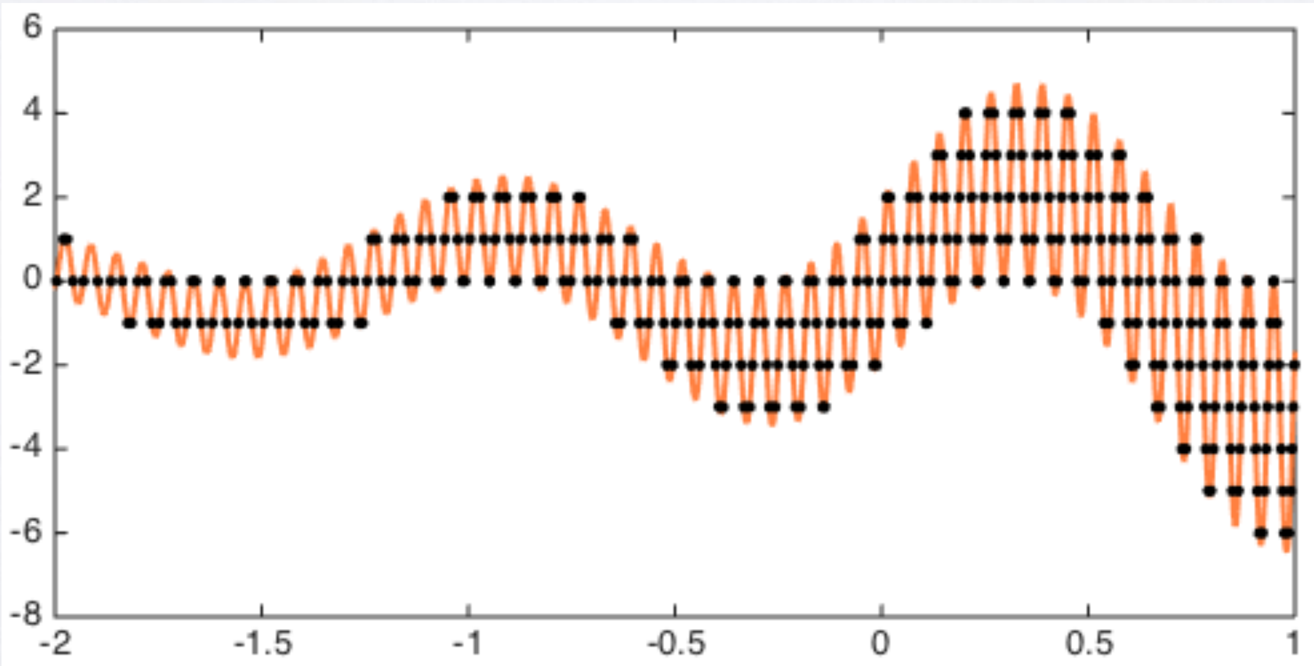
Matrix determinant

$$p(x) = \pm\det\left(A - xI\right)$$

characteristic polynomial of A

Identity matrix

A tiger's tail



OPEN PROBLEM

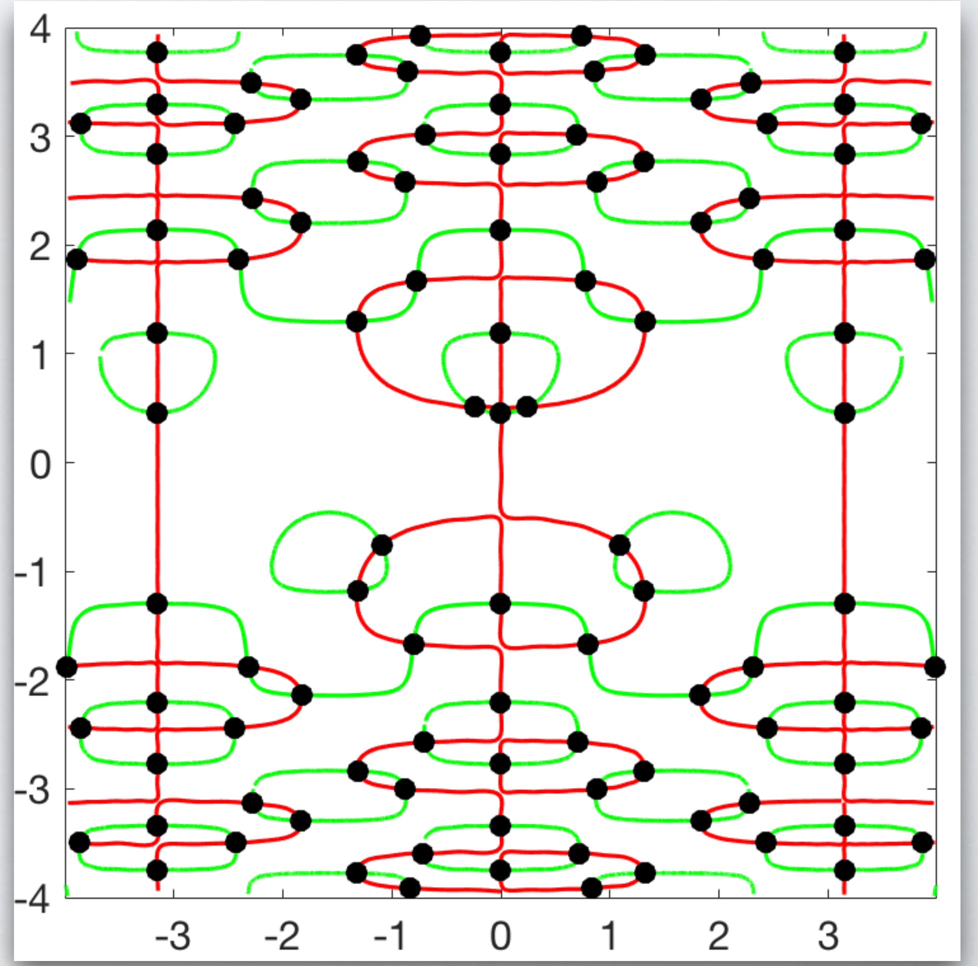
Let $p(x,y)$ be a degree (n,n) polynomial. Construct $n \times n$ matrices A, B, and C such that

$$p(x,y) = \det(A + xB + yC).$$

Compare to: $p(x) = \pm\det\left(A - xI\right)$

Need it to solve:

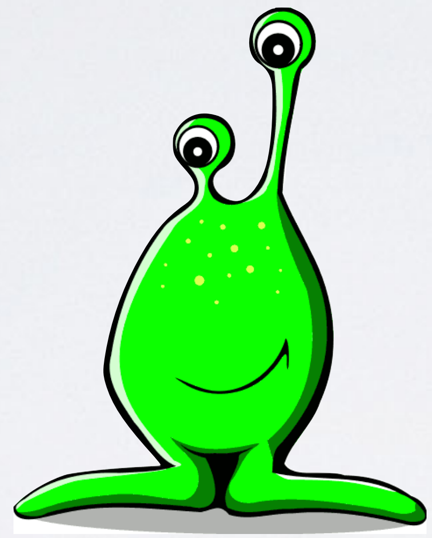
$$p(x,y) = q(x,y) = 0$$



1965: THE FAST FOURIER TRANSFORM

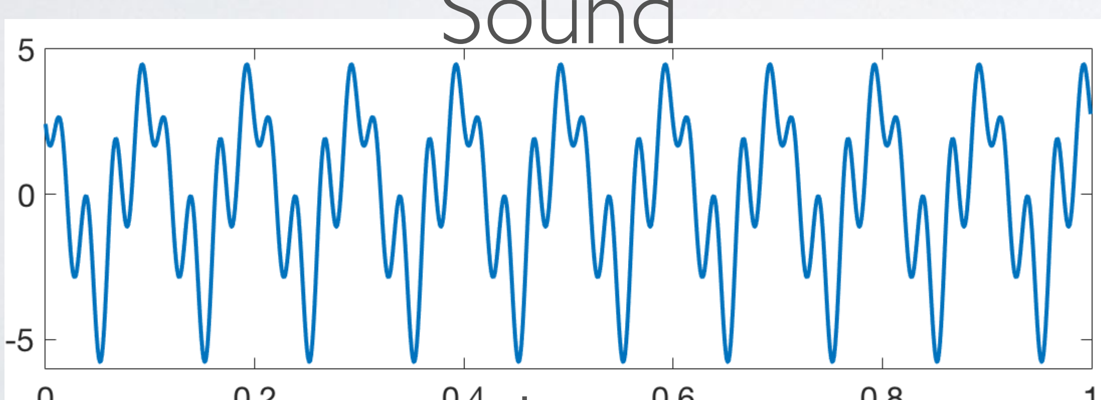
"Mozart could listen to music just once and then write it down from memory without any mistakes" [Vernon, 1996]

A simple example:



sound

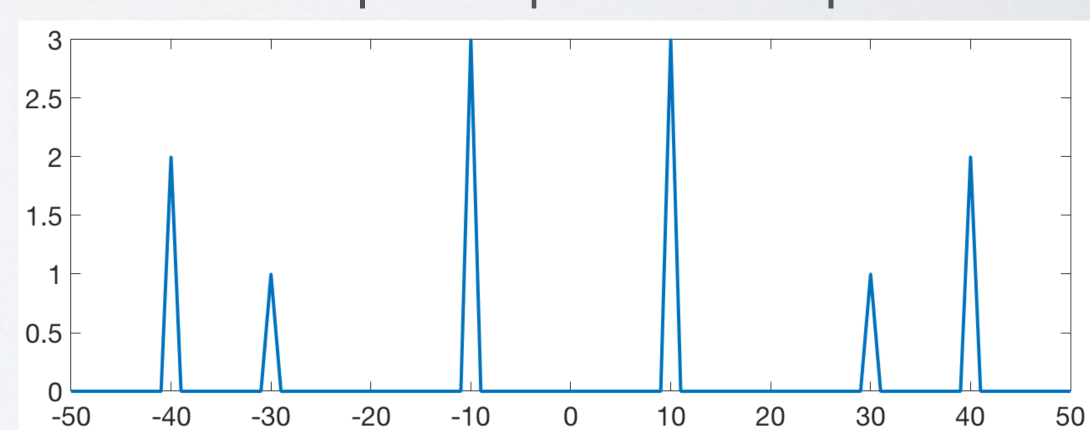
Sound



time

FFT

|Frequencies|



Hz

$$\text{sound}(t) = 3\cos(2\pi 10t + 0.2) + \cos(2\pi 30t - 0.3) + 2\cos(2\pi 40t + 2.4)$$

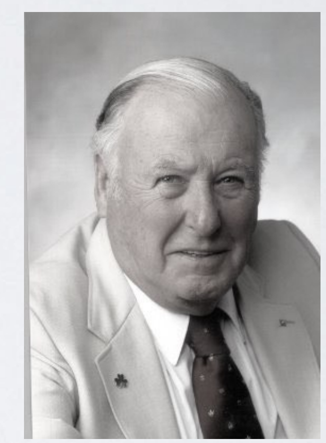
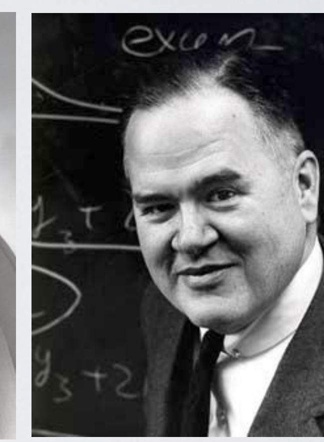
HOW DOES IT WORK?

Given equally spaced samples $f(0/n), f(1/n), \ldots, f((n-1)/n)$, find a_k so that

$$f(j/n) = \sum_{k=-n/2}^{n/2-1} a_k e^{2\pi i k (j/n)}, \qquad 0 \leq j \leq n-1.$$

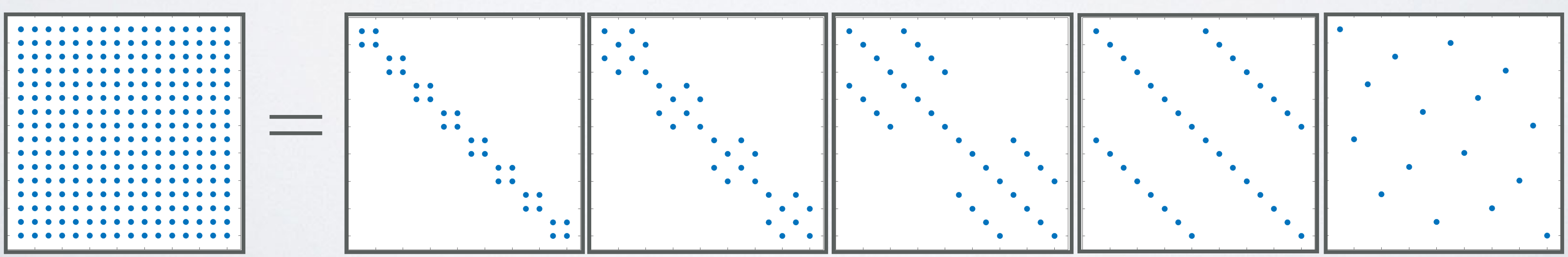
Fourier series

$$\begin{pmatrix} f(0/n) \\ \vdots \\ f((n-1)/n) \end{pmatrix} = F \begin{pmatrix} a_{-n/2} \\ \vdots \\ a_{n/2-1} \end{pmatrix}, \qquad F_{jk} = e^{2\pi i k (j/n)}$$

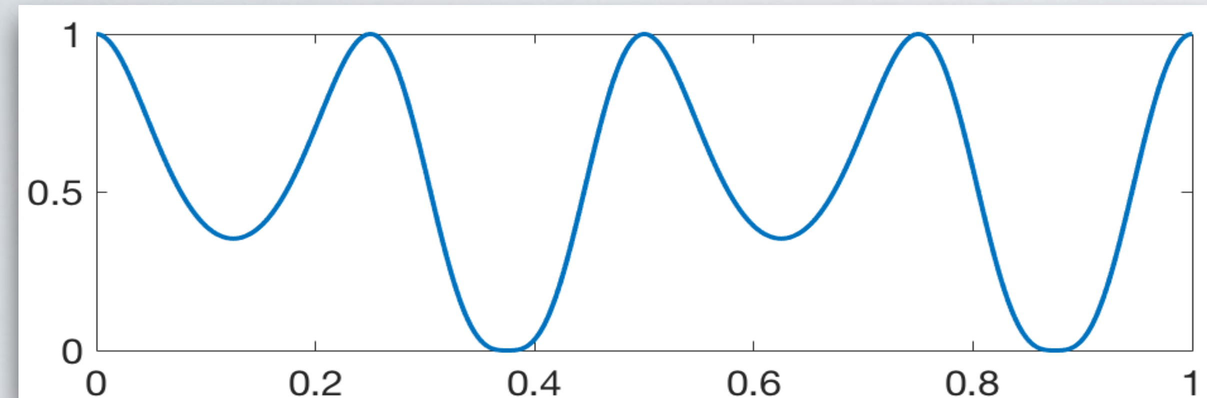
Cooley Tukey

F has a sparse factorization. For n = 16 we have

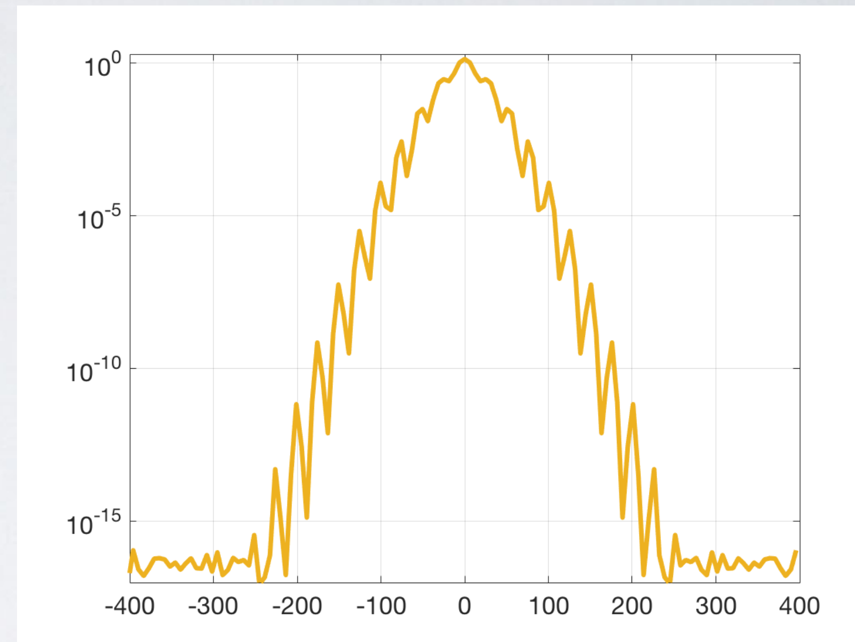


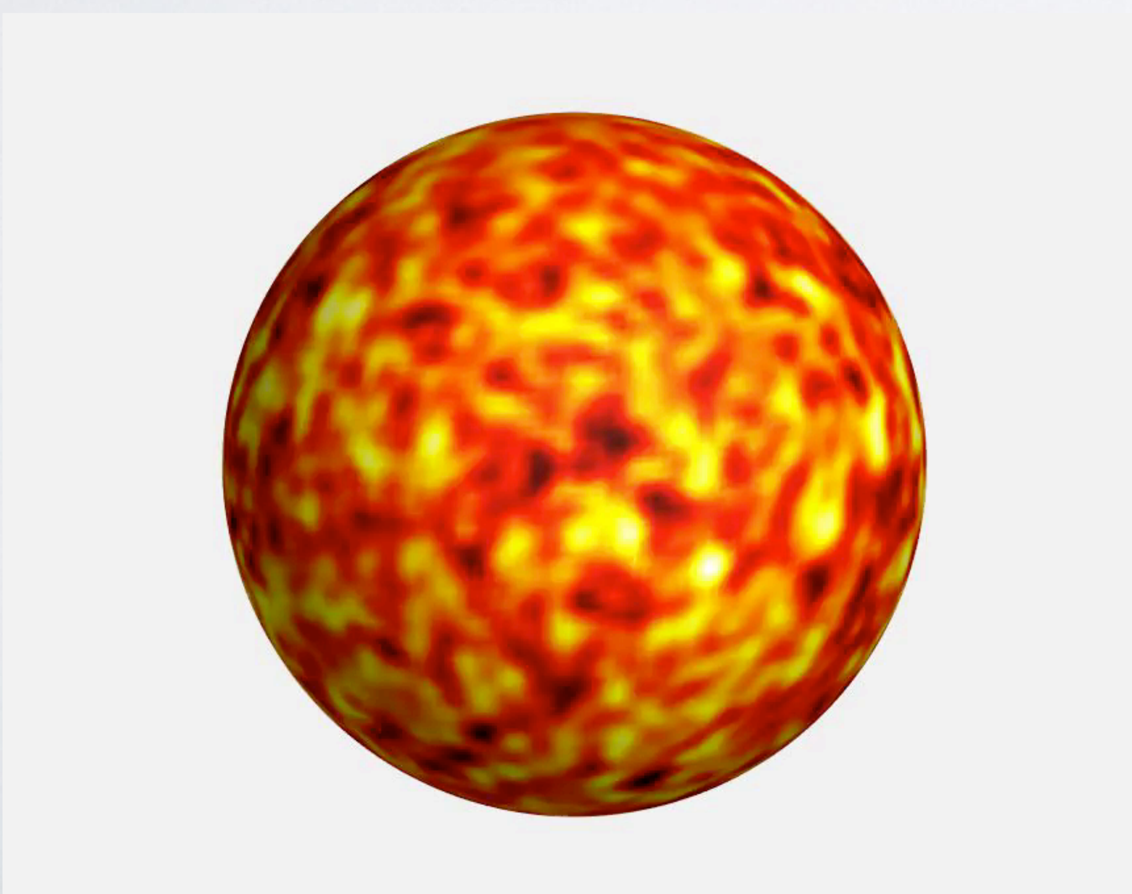
HOW DO I USE IT?

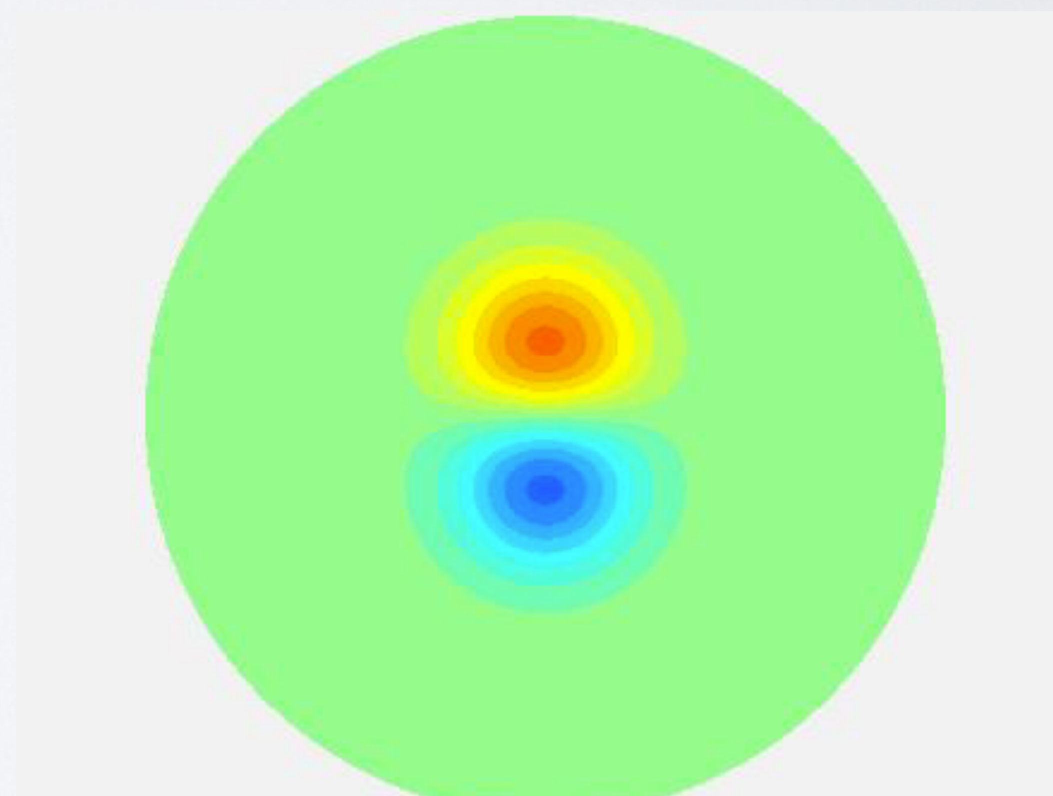
An automatic way to tell us how “complicated” a function is.



FFT





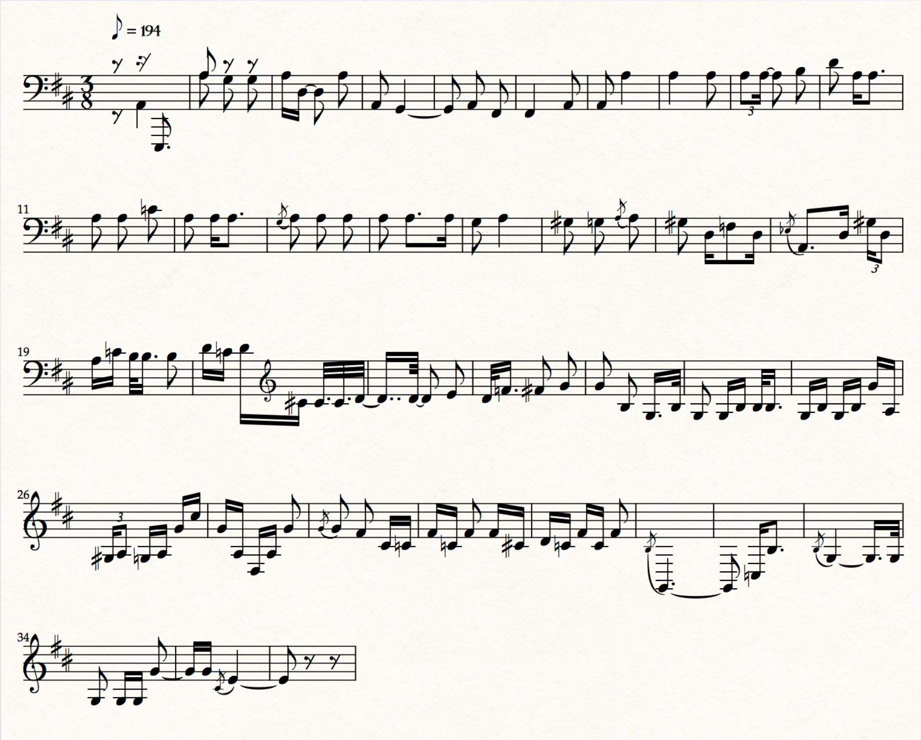


OPEN PROBLEM

Let everyone be a Mozart

An example with chords: Eight  playing

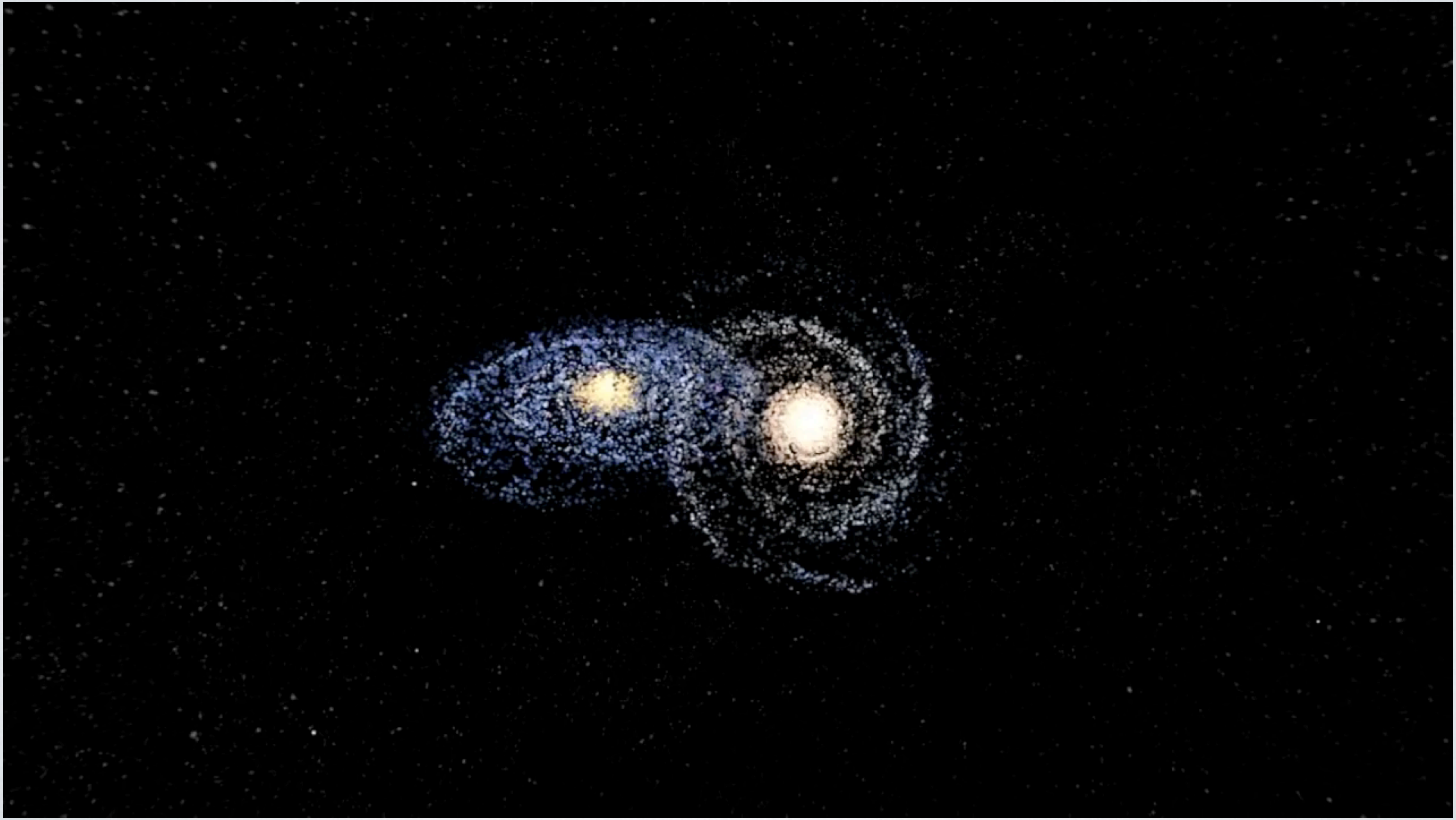
My sheet music for cellos



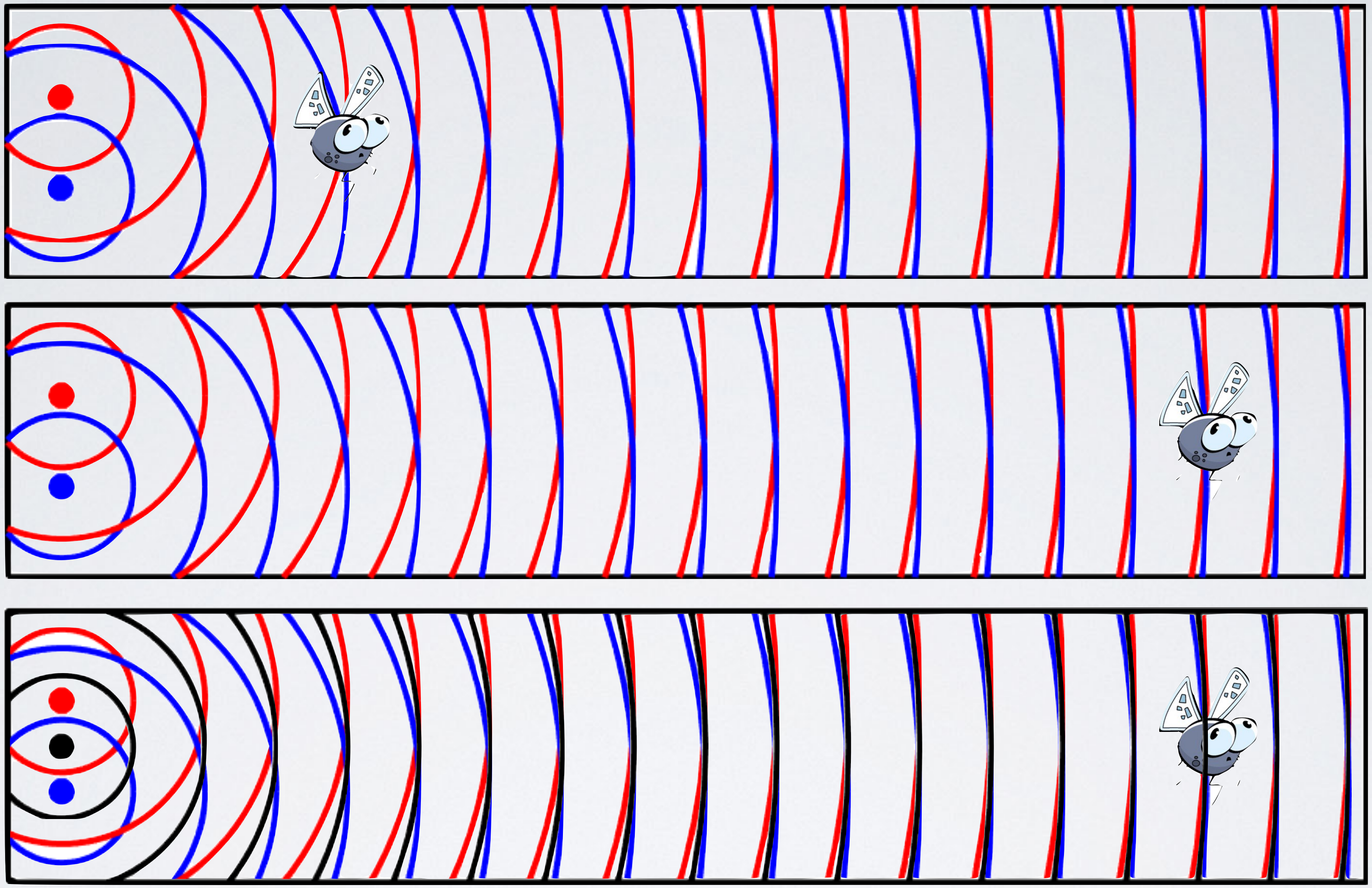
Play back

1987: THE FAST MULTIPOLE METHOD

In 4 billion years time…



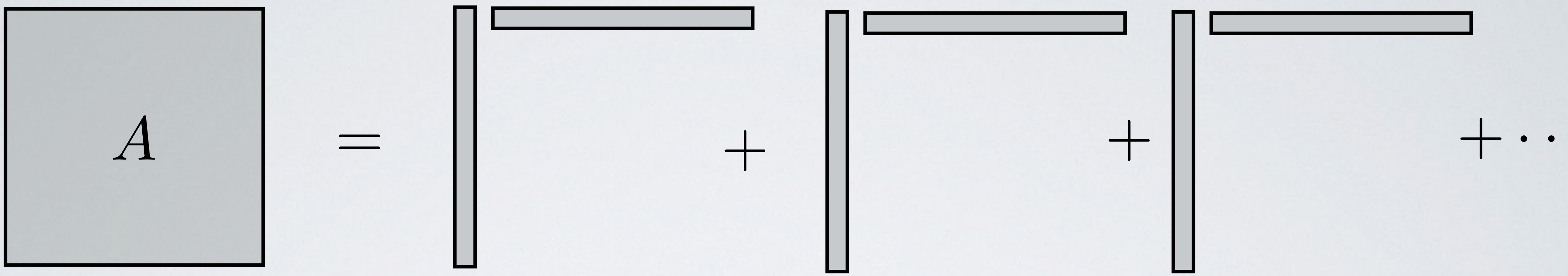
HOW DOES IT WORK?



Rokhlin

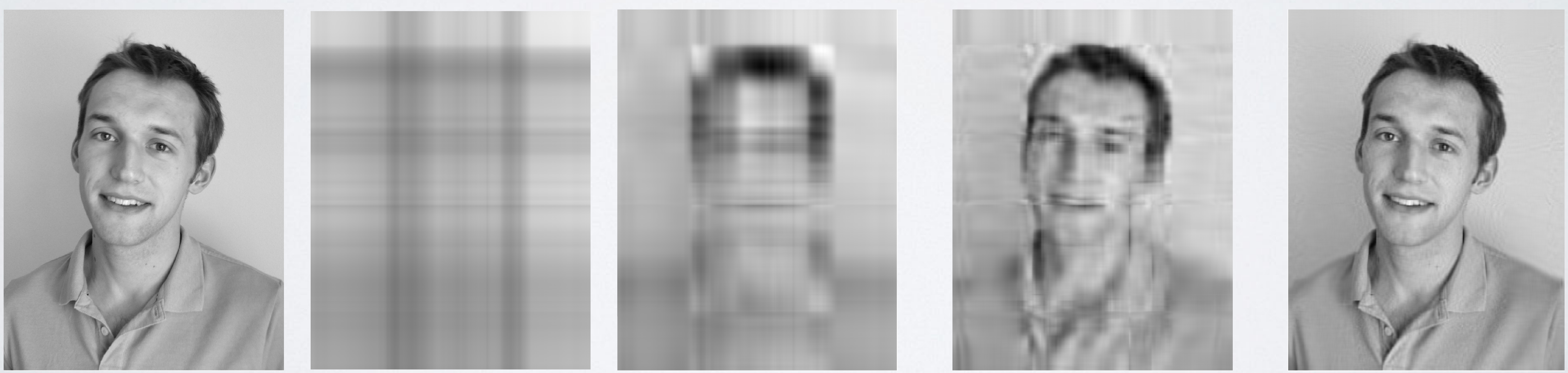
Greengard

HOW DO I USE IT?



Full rank form Low rank form

The SVD gives the best low rank approximations:

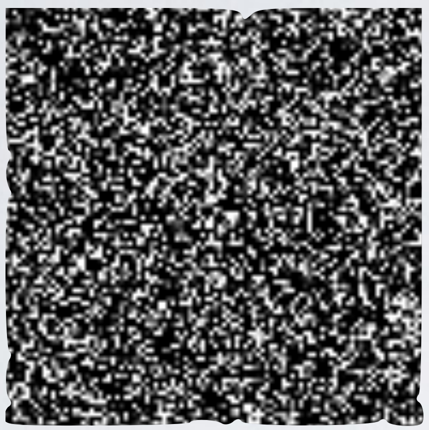


Original rank 1 rank 3 rank 10 rank 50

The low rank format saves computational time and storage costs

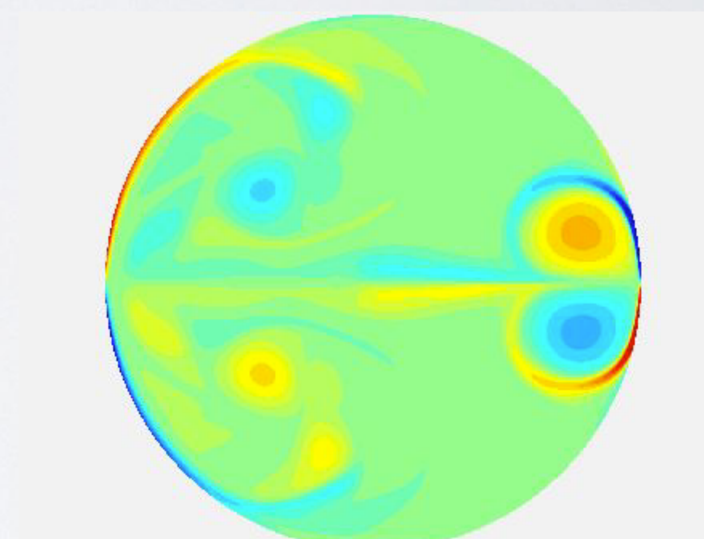
OPEN PROBLEM

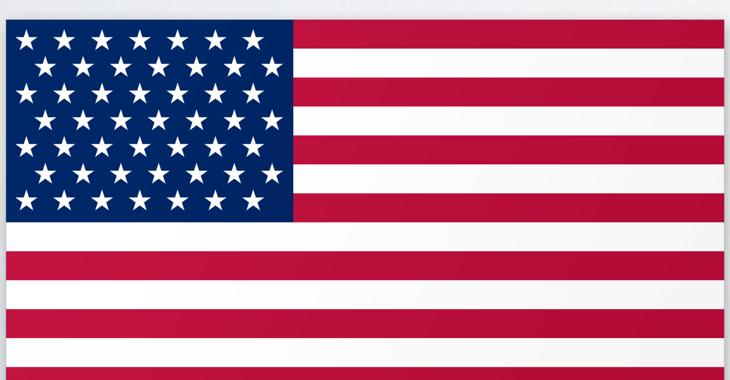
Why are so many matrices/functions in practice of low rank?



A random matrix is of full rank
so "average" matrices are not…

…but, these are of low rank.





Even the American flag is of low rank!

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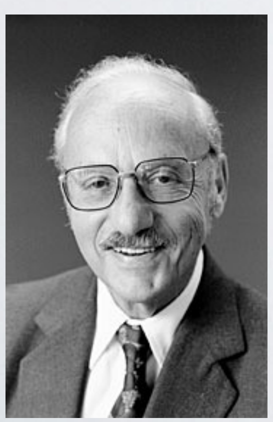
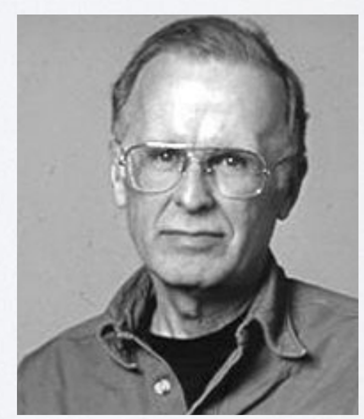

Dantzig


Neumann


Hestenes


Householder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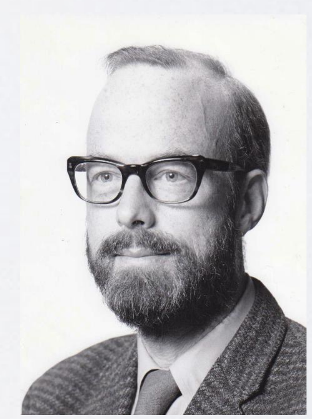
Hoare


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THANK YOU

What will be the top 10 algorithms of this century?

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