

Adams-Bashforth-Moulton Method

9.6 Predictor-Corrector Methods

The methods of Euler, Heun, Taylor, and Runge-Kutta are called *single-step methods* because they use only the information from one previous point to compute the successive point; that is, only the initial point (t_0, y_0) is used to compute (t_1, y_1) , and in general, y_k is needed to compute y_{k+1} . After several points have been found, it is feasible to use several prior points in the calculation. For illustration, we develop the Adams-Bashforth four-step method, which requires y_{k-3} , y_{k-2} , y_{k-1} , and y_k in the calculation of y_{k+1} . This method is not self-starting; four initial points (t_0, y_0) , (t_1, y_1) , (t_2, y_2) , and (t_3, y_3) , must be given in advance in order to generate the points $\{(t_k, y_k) : k \geq 4\}$ (this can be done with one of the methods from the previous sections).

A desirable feature of a multistep method is that the local truncation error (L.T.E.) can be determined and a correction term can be included, which improves the accuracy of the answer at each step. Also, it is possible to determine if the step size is small enough to obtain an accurate value for y_{k+1} , yet large enough so that unnecessary and time-consuming calculations are eliminated. Using the combinations of a predictor and corrector requires only two function evaluations of $f(t, y)$ per step.

Adams-Bashforth-Moulton Method

The Adams-Bashforth-Moulton predictor-corrector method is a multistep method derived from the fundamental theorem of calculus:

$$(1) \quad y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(t, y(t)) dt.$$

The predictor uses the Lagrange polynomial approximation for $f(t, y(t))$ based on the points (t_{k-3}, f_{k-3}) , (t_{k-2}, f_{k-2}) , (t_{k-1}, f_{k-1}) , and (t_k, f_k) . It is integrated over the interval $[t_k, t_{k+1}]$ in (1). This process produces the Adams-Bashforth predictor:

$$(2) \quad p_{k+1} = y_k + \frac{h}{24}(-9f_{k-3} + 37f_{k-2} - 59f_{k-1} + 55f_k).$$

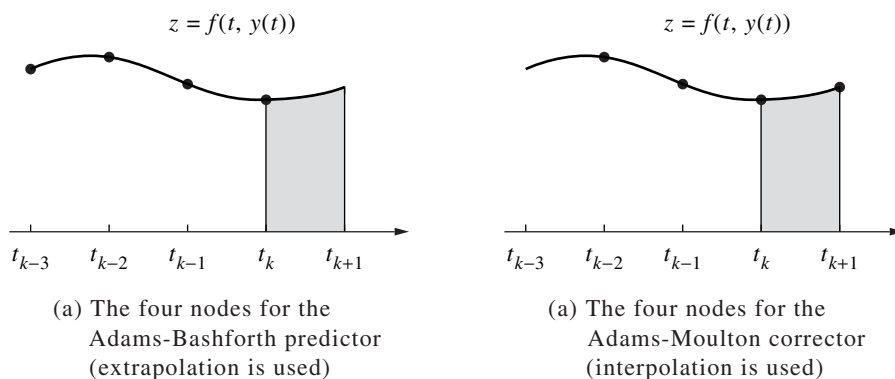


Figure 9.10 Integration over $[t_k, t_{k+1}]$ in the Adams-Bashforth method.

The corrector is developed similarly. The value p_{k+1} just computed can now be used. A second Lagrange polynomial for $f(t, y(t))$ is constructed, which is based on the points (t_{k-2}, f_{k-2}) , (t_{k-1}, f_{k-1}) , (t_k, f_k) , and the new point $(t_{k+1}, f_{k+1}) = (t_{k+1}, f(t_{k+1}, p_{k+1}))$. This polynomial is then integrated over $[t_k, t_{k+1}]$, producing the Adams-Moulton corrector:

$$(3) \quad y_{k+1} = y_k + \frac{h}{24}(f_{k-2} - 5f_{k-1} + 19f_k + 9f_{k+1}).$$

Figure 9.10 shows the nodes for the Lagrange polynomials that are used in developing formulas (2) and (3), respectively.

Error Estimation and Correction

The error terms for the numerical integration formulas used to obtain both the predictor and corrector are of the order $O(h^5)$. The L.T.E. for formulas (2) and (3) are

$$(4) \quad y(t_{k+1}) - p_{k+1} = \frac{251}{720}y^{(5)}(c_{k+1})h^5 \quad (\text{L.T.E. for the predictor}),$$

$$(5) \quad y(t_{k+1}) - y_{k+1} = \frac{-19}{720}y^{(5)}(d_{k+1})h^5 \quad (\text{L.T.E. for the corrector}).$$

Suppose that h is small and $y^{(5)}(t)$ is nearly constant over the interval; then the terms involving the fifth derivative in (4) and (5) can be eliminated, and the result is

$$(6) \quad y(t_{k+1}) - y_{k+1} \approx \frac{-19}{270}(y_{k+1} - p_{k+1}).$$

The importance of the predictor-corrector method should now be evident. Formula (6) gives an approximate error estimate based on the two computed values p_{k+1} and y_{k+1} and does not use $y^{(5)}(t)$.

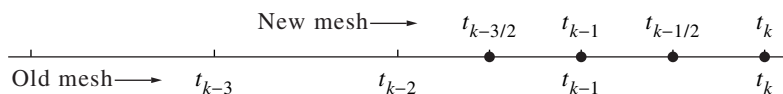


Figure 9.11 Reduction of the step size to $h/2$ in an adaptive method.

Practical Considerations

The corrector (3) used the approximation $f_{k+1} \approx f(t_{k+1}, p_{k+1})$ in the calculation of y_{k+1} . Since y_{k+1} is also an estimate for $y(t_{k+1})$, it could be used in the corrector (3) to generate a new approximation for f_{k+1} , which in turn will generate a new value for y_{k+1} . However, when this iteration on the corrector is continued, it will converge to a fixed point of (3) rather than the differential equation. It is more efficient to reduce the step size if more accuracy is needed.

Formula (6) can be used to determine when to change the step size. Although elaborate methods are available, we show how to reduce the step size to $h/2$ or increase it to $2h$. Let $\text{RelErr} = 5 \times 10^{-6}$ be our relative error criterion, and let $\text{Small} = 10^{-5}$.

$$(7) \quad \text{If } \frac{19}{270} \frac{|y_{k+1} - p_{k+1}|}{|y_{k+1}| + \text{Small}} > \text{RelErr}, \quad \text{then set } h = \frac{h}{2}.$$

$$(8) \quad \text{If } \frac{19}{270} \frac{|y_{k+1} - p_{k+1}|}{|y_{k+1}| + \text{Small}} < \frac{\text{RelErr}}{100}, \quad \text{then set } h = 2h.$$

When the predicted and corrected values do not agree to five significant digits, then (7) reduces the step size. If they agree to seven or more significant digits, then (8) increases the step size. Fine-tuning of these parameters should be made to suit your particular computer.

Reducing the step size requires four new starting values. Interpolation of $f(t, y(t))$ with a fourth-degree polynomial is used to supply the missing values that bisect the intervals $[t_{k-2}, t_{k-1}]$ and $[t_{k-1}, t_k]$. The four mesh points $t_{k-3/2}$, t_{k-1} , $t_{k-1/2}$, and t_k used in the successive calculations are shown in Figure 9.11.

The interpolation formulas needed to obtain the new starting values for the step size $h/2$ are

$$(9) \quad \begin{aligned} f_{k-1/2} &= \frac{-5f_{k-4} + 28f_{k-3} - 70f_{k-2} + 140f_{k-1} + 35f_k}{128}, \\ f_{k-3/2} &= \frac{3f_{k-4} - 20f_{k-3} + 90f_{k-2} + 60f_{k-1} - 5f_k}{128}. \end{aligned}$$

Increasing the step size is an easier task. Seven prior points are needed to double the step size. The four new points are obtained by omitting every second one, as shown in Figure 9.12.

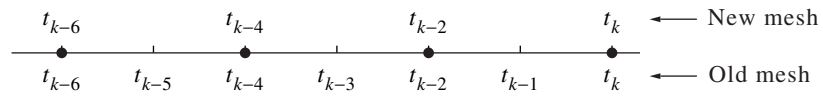


Figure 9.12 Increasing the step size to $2h$ in an adaptive method.

Numerical Methods Using Matlab, 4th Edition, 2004

John H. Mathews and Kurtis K. Fink

ISBN: 0-13-065248-2

Prentice-Hall Inc.

Upper Saddle River, New Jersey, USA

<http://vig.prenhall.com/>

NUMERICAL METHODS USING MATLAB

FOURTH EDITION



JOHN H. MATHEWS • KURTIS D. FINK