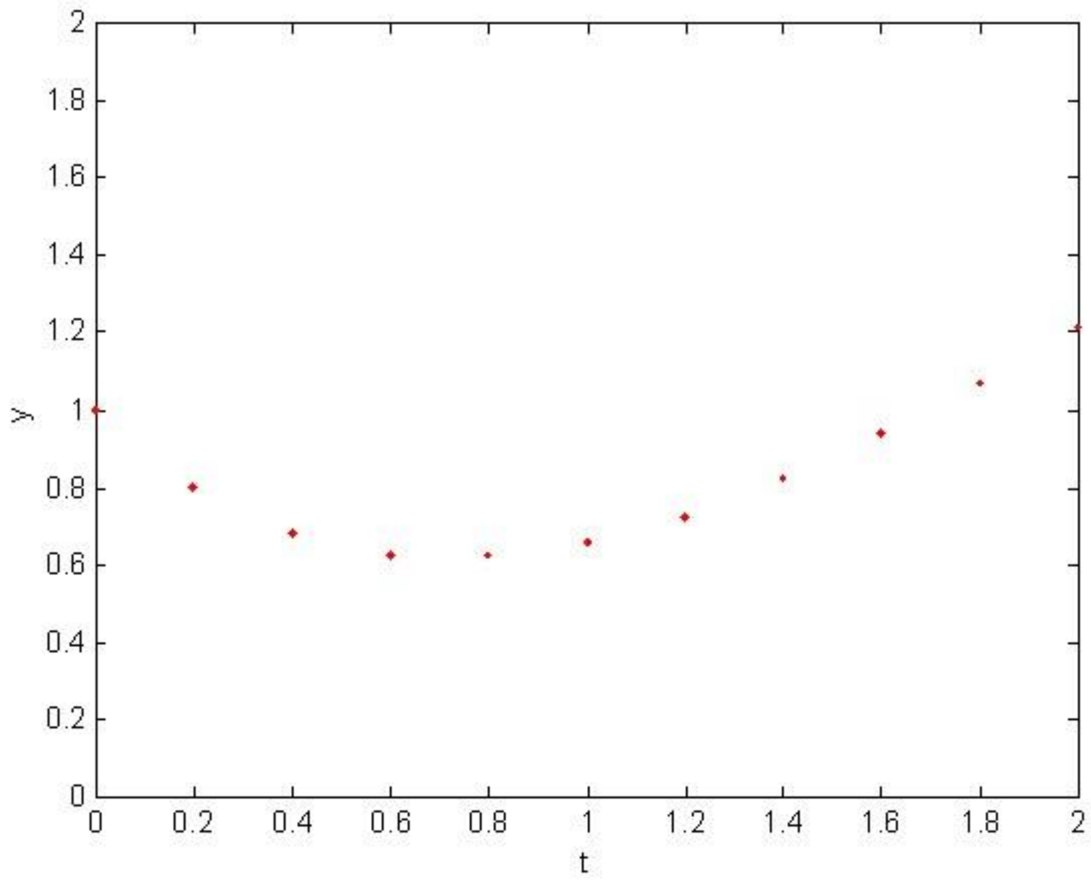
Numerical solution of ordinary differential equations

Trajectory

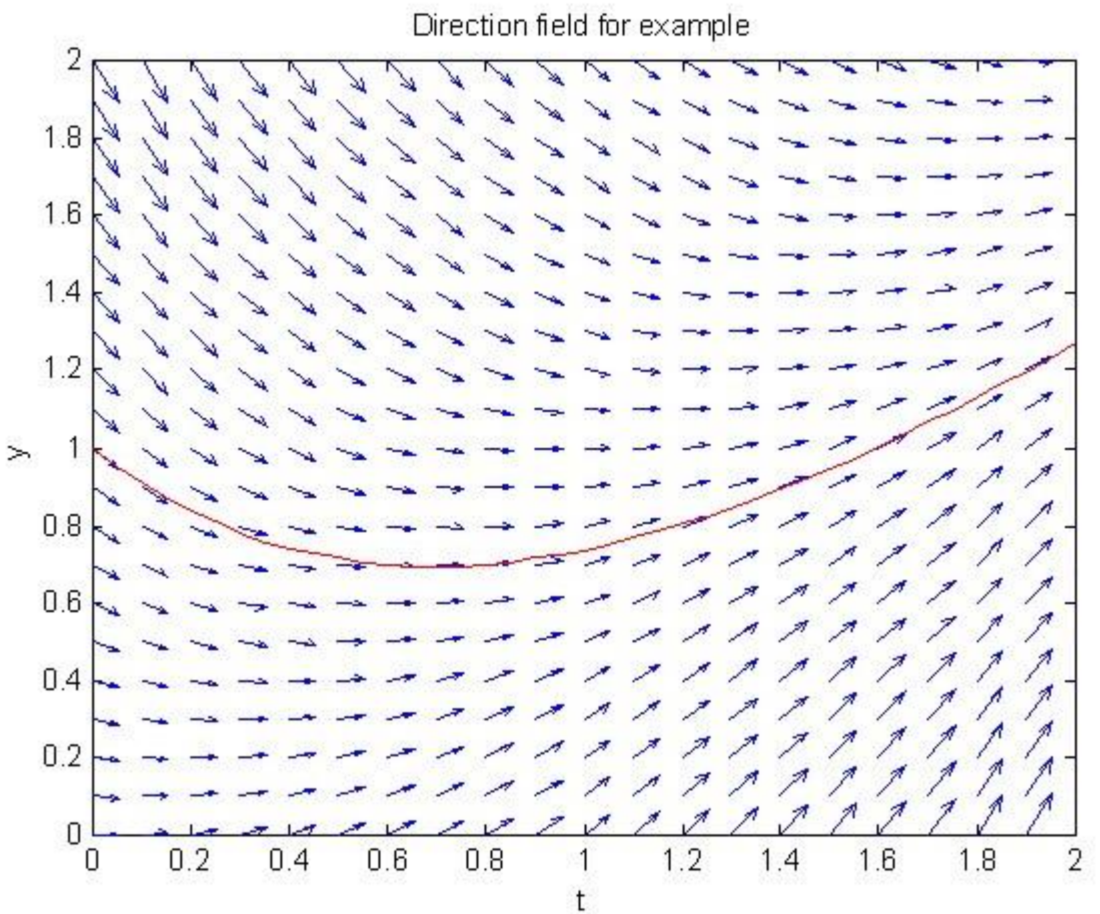
$$\frac{dy}{dt} = f(t, y)$$

- One of the task for ODE solvers is to find the trajectory starting from initial conditions



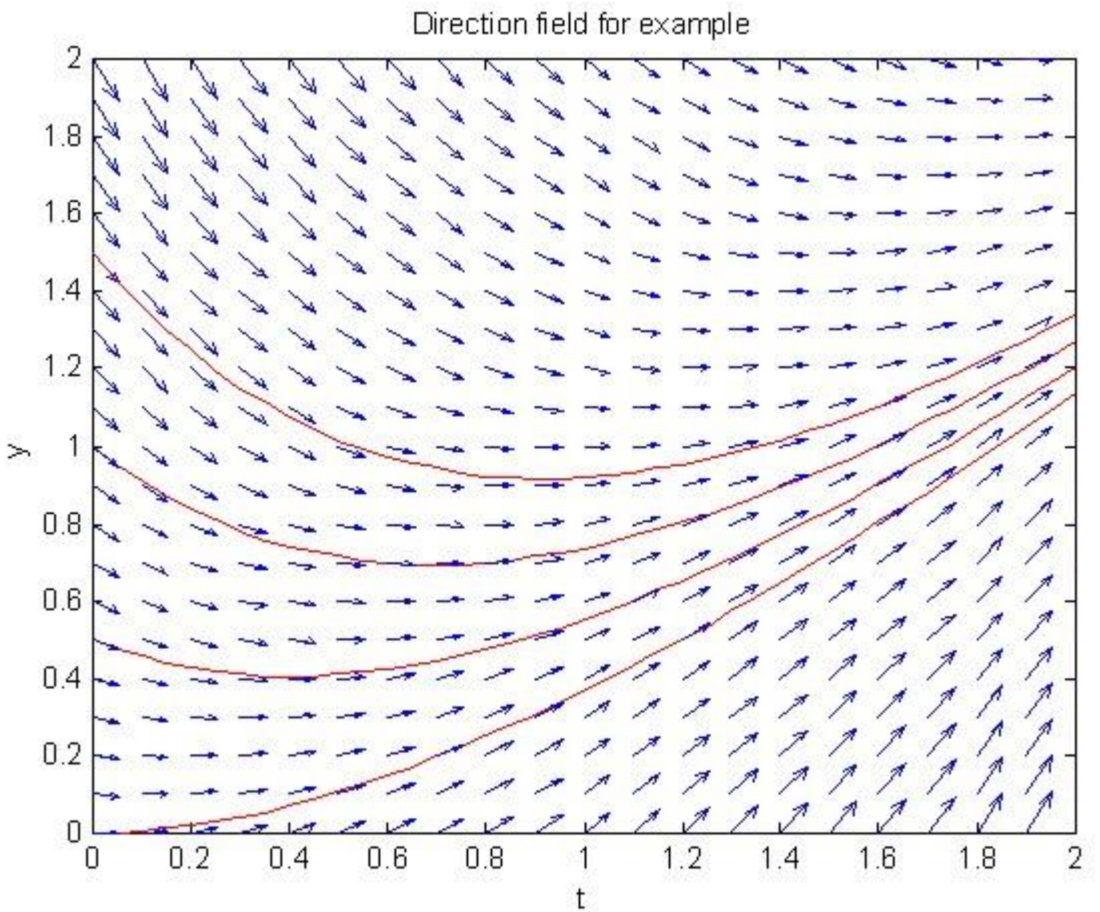
True trajectory

- There are infinitely many possible trajectories
- The derivatives “show the correct path”



Note: some images, slides from RN Shorten, D. Leith: https://slideplayer.com/slide/4594068/)

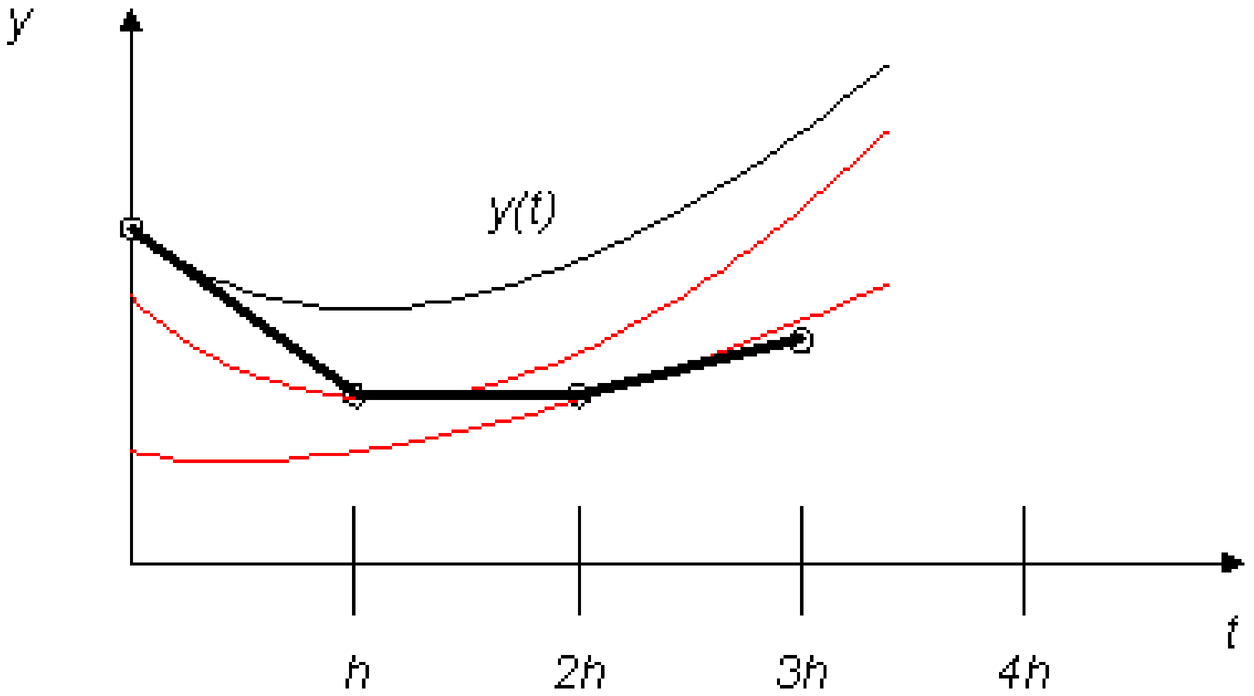
True trajectory



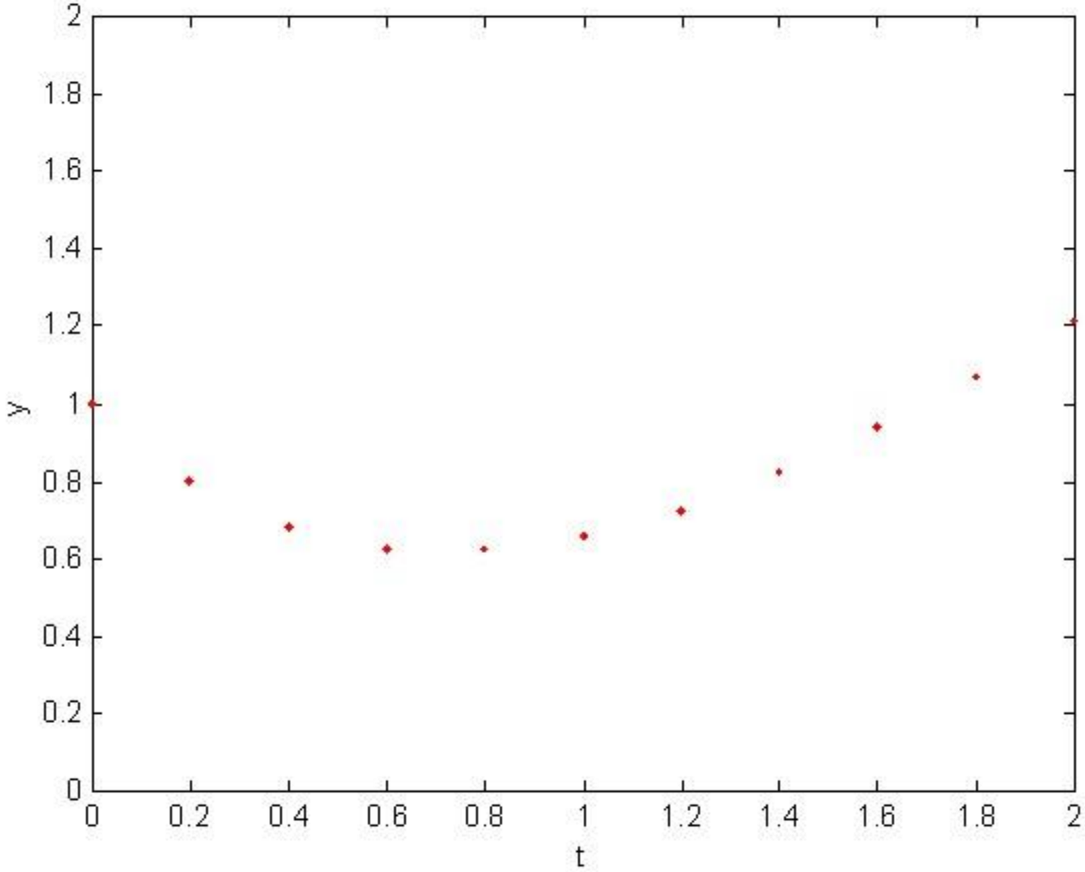
- We can calculate derivatives at all *(t,y)* points
- It depends on *y*, too, not only on *t*!

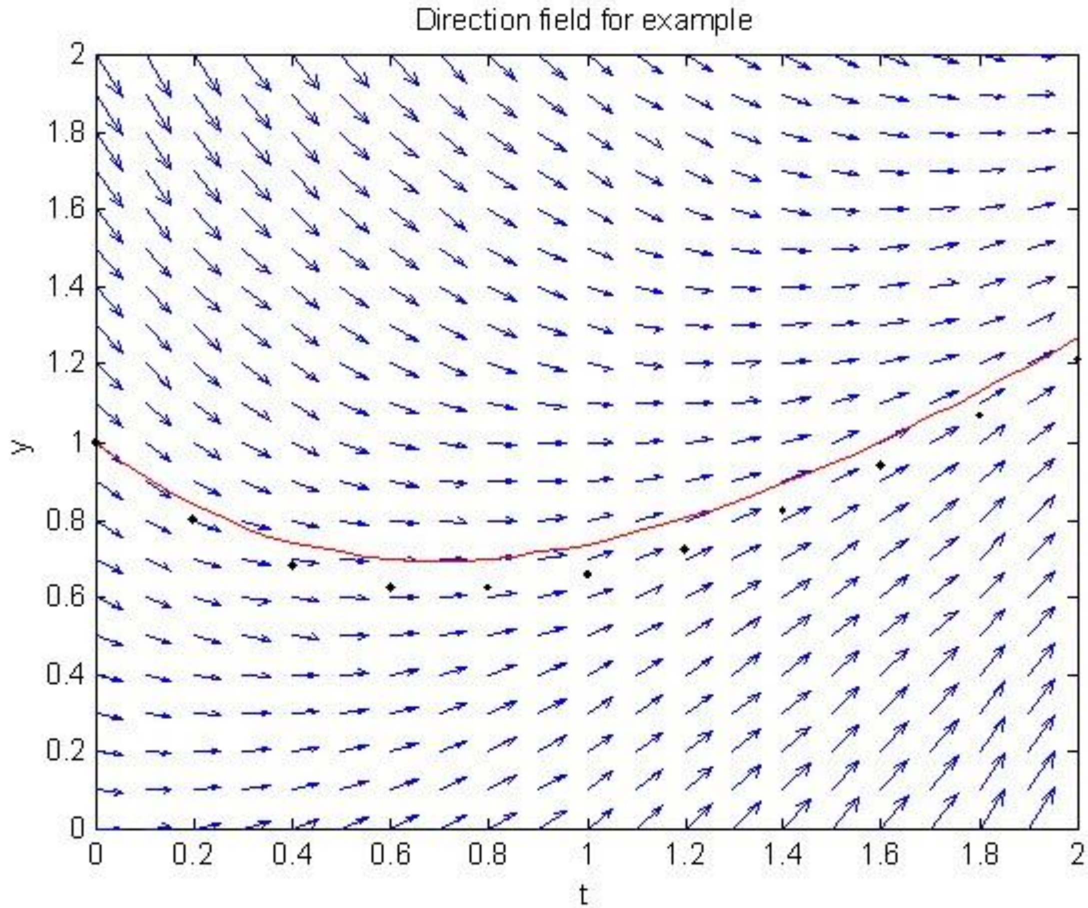
Euler's method

- Using derivatives at the actual points to estimate solution
- Errors accumulate

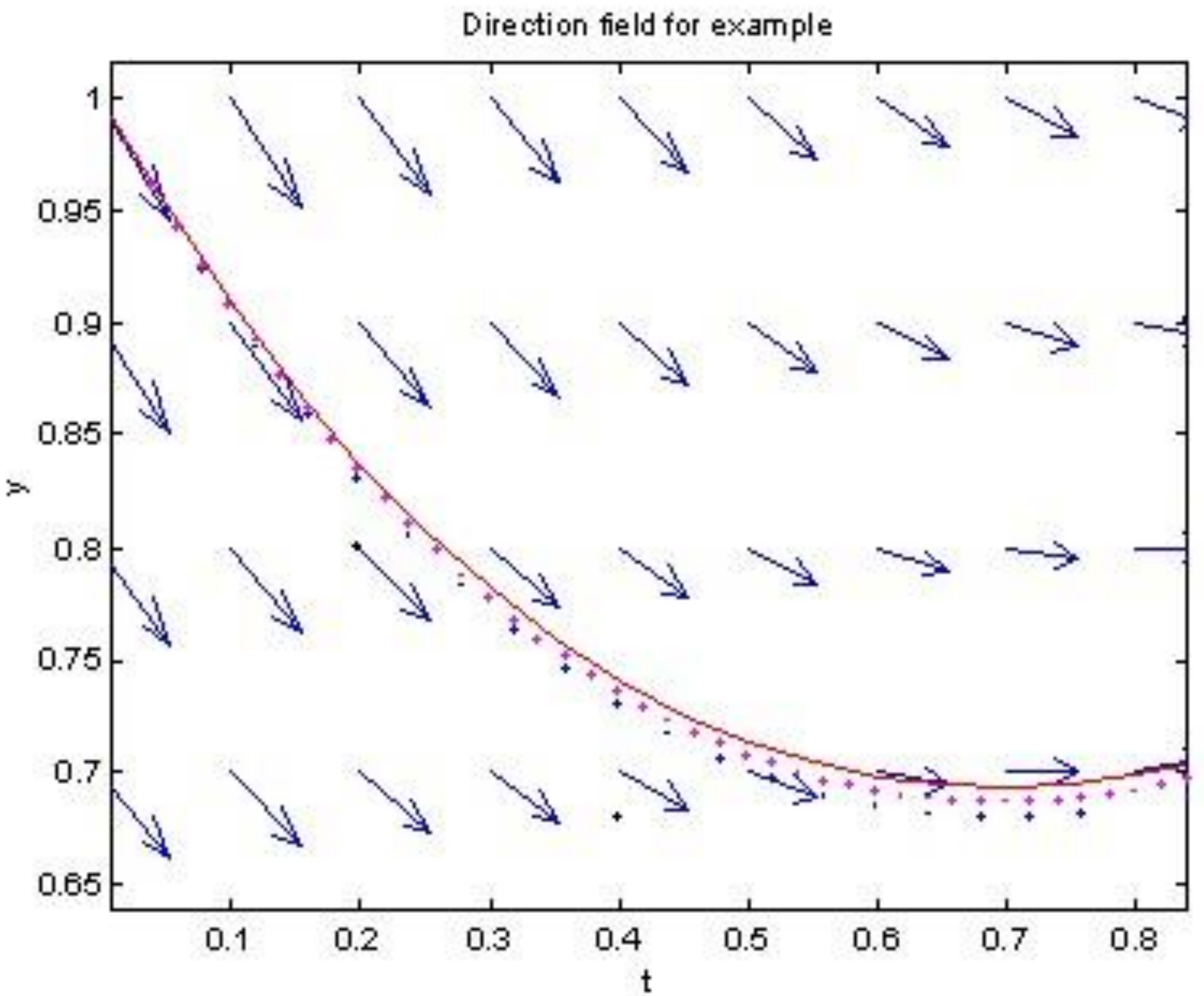


Discrete solution: sampling the true trajectory




Direction field for example
y
t
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Smaller stepsize: closer to true solution



Multi-step concept

$$\frac{dy}{dt} = f(t, y)$$

- Euler, RK4, … all have the form:

$$\hat{y}_{k+1} = K\hat{y}_k + g(t, \hat{y}_k)$$

- Prediction depends **only on previous value**
- Multi-step methods attempt to gain efficiency by keeping and using the information from previous steps rather than discarding it. Consequently, multistep methods refer to **several previous points** and derivative values.

Adams-Bashforth Moulton method

$$\frac{dy}{dt} = f(t, y)$$

- The ABM predictor-corrector method is a multi-step method. It is obtained by approximating the integral in the formula

$$y(t_{k+1}) - y(t_k) = \int_{t_k}^{t_{k+1}} f(t, y(t))dt$$

by an interpolation polynomial of third degree. This polynomial is chosen so that is passes through the points

$$(t_{k-n}, f_{k-n}), \ldots, n = 0,1,2$$

This produces an ABM prediction

$$p(t_{k+1})$$

Adams-Bashforth Moulton method

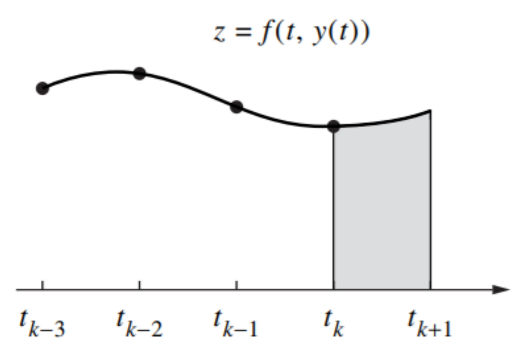
- Having obtained the prediction, a second polynomial is constructed to fit the points

$$(t_{k-n}, f_{k-n}), \ldots, n = 0,1,2$$

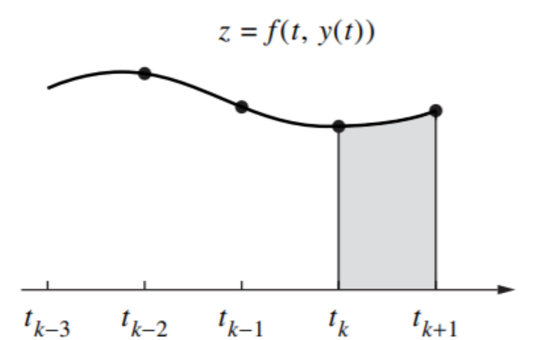
- and the point

$$(t_{k+1}, p(t_{k+1}))$$

- This polynomial is then integrated to obtain the final prediction at time index *k+1.*



(a) The four nodes for the Adams-Bashforth predictor (extrapolation is used)



(a) The four nodes for the Adams-Moulton corrector (interpolation is used)

Adams-Bashforth Moulton method

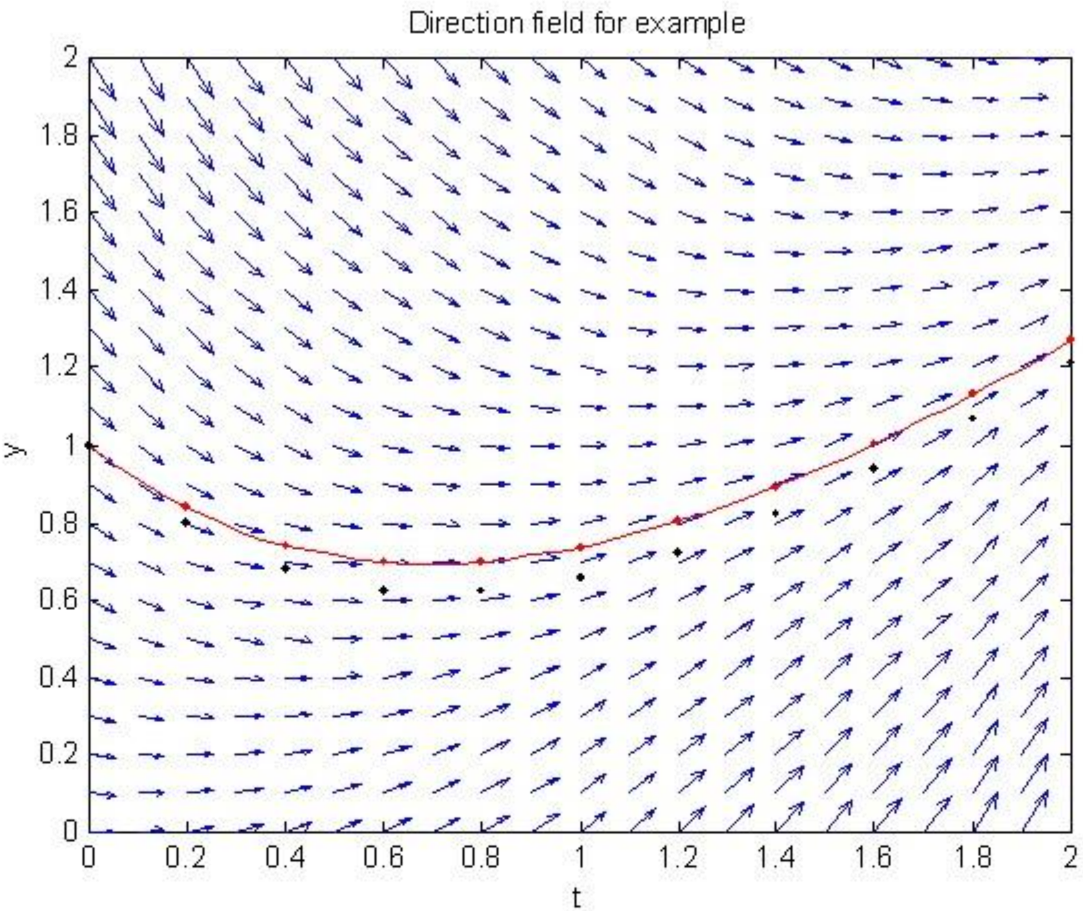
- The ABM formula's are:

$$p(t_{k+1}) = \hat{y}(t_k) + \frac{h}{24}(-9f_{k-3} + 37f_{k-2} - 59f_{k-1} + 55f_k)$$

$$\hat{y}(t_{k+1}) = \hat{y}(t_k) + \frac{h}{24}(9f_{k-2} - 5f_{k-1} + 19f_k + f_{k+1})$$

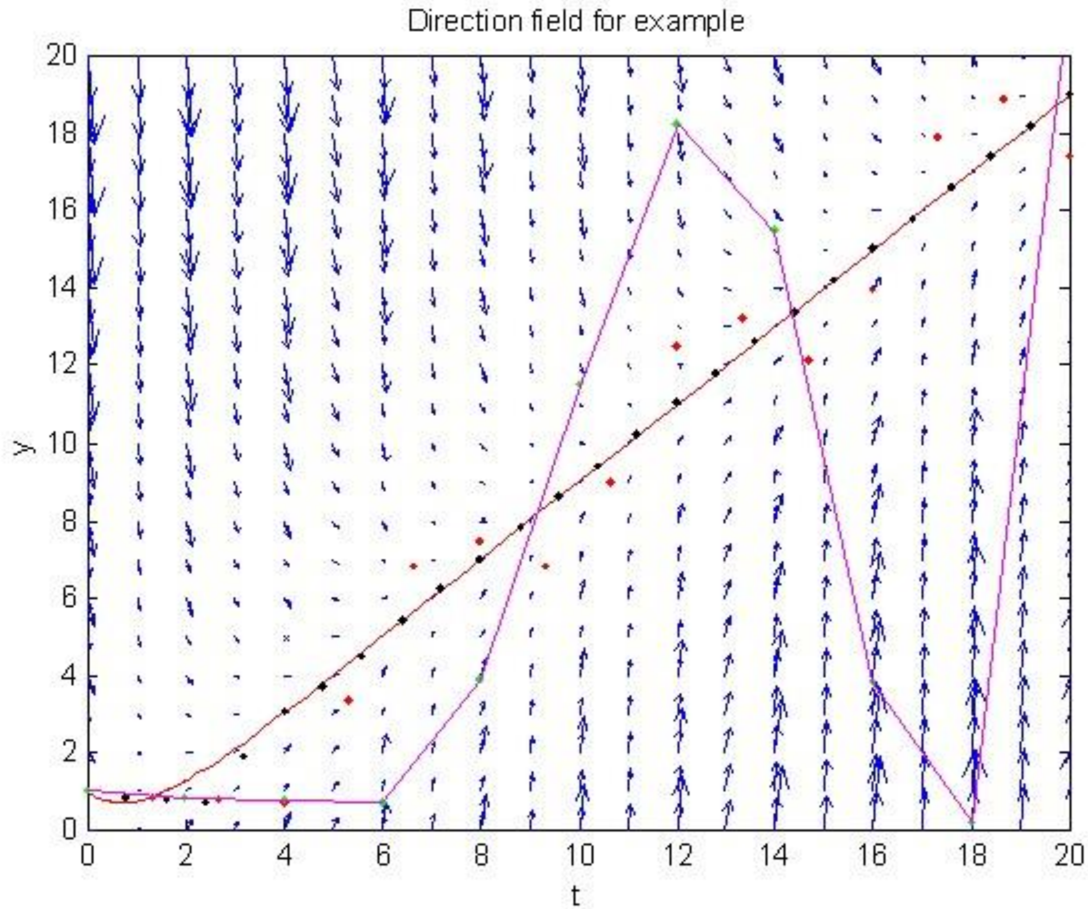
- The advantage of this technique is that the difference between the predictor and corrector gives an estimate of the truncation error.

ABM: more precise



ABM: but can be instable

- There is a range for step size where the method is instable



Optimal step size

- Runge-Kutta: step size error can be calculated (make one step with *h->y_1* and one with *h/2->y_2*): $\Delta = y_2 - y_1$
- RK4 has 5-th order error, so if the expected error is Δ_0, then the optimal step is

$$h_0 = h \left|\frac{\Delta_0}{\Delta}\right|^{0.2}$$

Stability

- Take this simple example:

$$y' = -cy \qquad c > 0$$

- Euler "explicit" solution:

$$y_{n+1} = y_n + hy'_n = (1 - ch)y_n$$

- If the step size is "too large" $h > 2/c$ than the solution $|y_n|$ goes to infinity instead of the true solution, 0.

- Example 2 equations:

$$u' = 998u + 1998v \qquad\qquad v' = -999u - 1999v$$

- With boundary condition: $u(0) = 1,\, v(0) = 0$
- With this variable transformation:

$$v = -y + z \qquad\qquad u = 2y - z$$

- the equations take the form:

$$v = -e^{x} + e^{-1000x} \qquad\qquad u = 2e^{-x} - e^{-1000x}$$

- Because of the second terms, stability (*h<2/c*) would require *h<2/1000.* But they are negligibly small, so because of the instability of equations very small steps are required.

Implicit Euler method

- Backwards Euler step:

$$y_{n+1} = y_n + h y'_{n+1}$$

- This is an “implicit equation since y_{n+1} appears on the right hand side, too.
- For our first example the equation can be reordered:

$$y' = -cy \qquad c > 0 \qquad\qquad y_{n+1} = \frac{y_n}{1 + ch}$$

- This is always stable without any restriction on *h*.
- The price to be paid for stability: solution of set of (linear) equations in **each step**.

Instability can be hidden

$$v = -e^{x} + e^{-1000x}$$

$$u = 2y - z$$

$$v = -y + z$$

$$u' = 998u + 1998v$$

$$v' = -999u - 1999v$$

$$u(0) = 1,\ v(0) = 0$$

$$u = 2e^{-x} - e^{-1000x}$$