Lotka-Volterra models

Numerical Simulation

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Contents

1 Introduction

The Lotka-Volterra model (LVM) is describing predator-prey like interactions and can be used to describe the behaviour of biological systems and neural networks. It can even be used more widely, if modifications are made in order to make it more realistic, more powerful to give solutions to some problems occurring in Physics or in other fields of science.

The motivation behind this project is to examine the basic LVM and its modifications until it gives satisfactory results and meet our criteria, because - as said before - it serves as a theoretical approach to various problems. Hence I would like to investigate the LVM in this work. I will also attempt to show where it can be used in plasma physics to describe a phenomena of turbulent flows.

To turn the previous words into coins the simulation and evaluation will be made step-by-step with the help of the following book [1] and answers, solutions to the arising questions will be found.

2 Lotka-Volterra models

To get started, the basic quantities should be introduced. Let

$$p(t) = \text{prey density}, \; P(t) = \text{predator density}. \tag{1}$$

These notations will be used throughout this work and a 4th-order Runge-Kutta explicit solver will be applied to get the results. Each run will cover the time interval between 0 and 500 and overall 10^5 time steps will be made.

2.1 LVM-I

2.1.1 Background

We assume in the absence of interactions, that the prey population p grows at a per-capita rate of a, which would lead to exponential growth:

$$\frac{dp(t)}{dt} = ap, \rightarrow p(t) = p(0)e^{at}. \tag{2}$$

Though, in reality it will not occur, because predators P will be able to eat more prey as p increase. The interaction rate between prey and predators can be introduced the following way, since it needs both to be present:

$$\text{Interaction rate} = bpP, \tag{3}$$

where b is describing the strength of the interaction.

From this we can get the first, most basic equation for the prey population:

$$\frac{dp(t)}{dt} = ap(t) - bp(t)P(t), \text{ (LVM-I for prey)}. \tag{4}$$

To get an equation for the predator population, we need to consider two things. First, predators will eat themselves if no prey present, therefore it is wise to introduce:

$$\left.\frac{dP(t)}{dt}\right|_{competition} = -mP(t), \tag{5}$$

where m is the per-capita mortality rate. The second consideration is, once a predator caught a prey, it has an efficiency (ε) to convert it into food. Therefore:

$$\frac{dP(t)}{dt} = \varepsilon bp(t)P(t) - mP(t), \text{ (LVM-I for predator)}. \tag{6}$$

In this simple model, the equilibrium values for the populations can be found easily and depend on the parameters:

$$p(t) = \frac{m}{\varepsilon b}, \text{ and } P(t) = \frac{a}{b}. \tag{7}$$

2.1.2 Results

To validate the obtained results, the same initial values and parameters were used as written in [1] (see Table 1).

Model	a	b	ε	m
LVM-I	0.2	0.1	1.0	0.1

Table 1: Parameters used in the simulation.

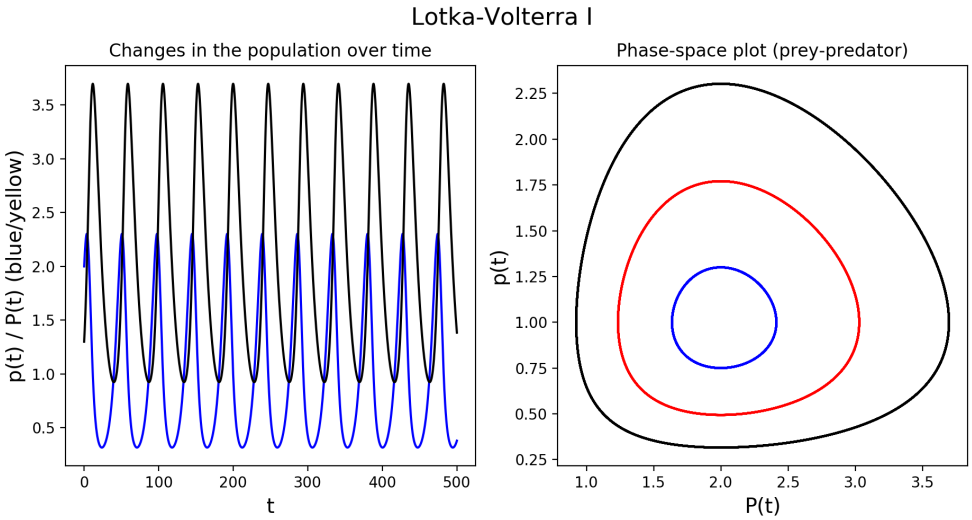

Figure 1: Left: changes in the prey population (blue) and the predator population (black), when initiated with $p(0) = 2.0$ and $P(0) = 1.3$. Right: phase-space plot using three different initial conditions for the populations (results marked with different colors for each).

It can be seen, that the amplitudes are the same, so that the prey and predator populations minimum and maximum values do not change in time and the minimum prey-maximum predator populations (and vica versa) are not occuring at the same time. The latter can be explained with the fact, that the growing rate in the prey and the declining rate in the predator populations are exponential, while the interaction rate is proportional to the joint probability. The exponentials are causing the shift, since these grow/decay over a specific time $\left(\propto \frac{1}{a}\right)/\left(\propto \frac{1}{m}\right)$ and not responding immediately to the changes.

One can also see from the phase-space plot, that the amplitudes strongly depends on the given initial values. Therefore it is not a very realistic model, we need further modifications.

The equilibrium state can be calculated with (7) and we can get the figure we expect (with the previous initial conditions):

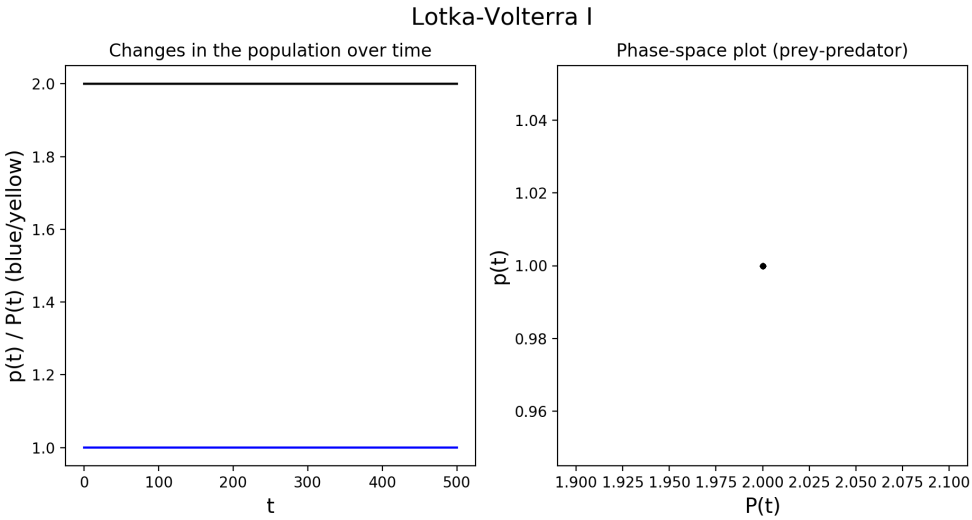

Figure 2: Equilibrium state, prey (blue), predator (black).

2.2 LVM-II

2.2.1 Background

The first step for modifying the previously presented LV-I is to introduce a maximum limit (K) for the prey population. We should take into account, that the supplies for the preys are also limited, so the exponential growth is to prevent. The governing equations in this case:

$$\frac{dp(t)}{dt} = ap(t)\left[1 - \frac{p(t)}{K}\right] - bp(t)P(t), \text{ (LVM-II for prey)}, \tag{8}$$

where K is an upper limit, so the carrying capacity for the prey population.

$$\frac{dP(t)}{dt} = \varepsilon bp(t)P(t) - mP(t), \text{ (LVM-II for predator)}. \tag{9}$$

The equilibrium values can be also given here, but it will not be as simple as before:

$$p(t) = \frac{m}{\varepsilon b}, \tag{10}$$

$$P(t) = \frac{a}{b}\left(1 - \frac{p(t)}{K}\right) = \frac{a}{b}\left(1 - \frac{m}{\varepsilon b} \cdot \frac{1}{K}\right). \tag{11}$$

So the equilibrium of the prey population (10) will determine the equilibrium of the predator population (11). One can also see, that this equilibrium will depend on the prey carrying capacity as a function $f\left(\frac{1}{K}\right)$.

2.2.2 Results

To validate the obtained results, the same initial values and parameters were used as written in [1] (see Table 2).

Model	a	b	ε	m	K
LVM-II	0.2	0.1	1.0	0.1	20

Table 2: Parameters used in the simulation.

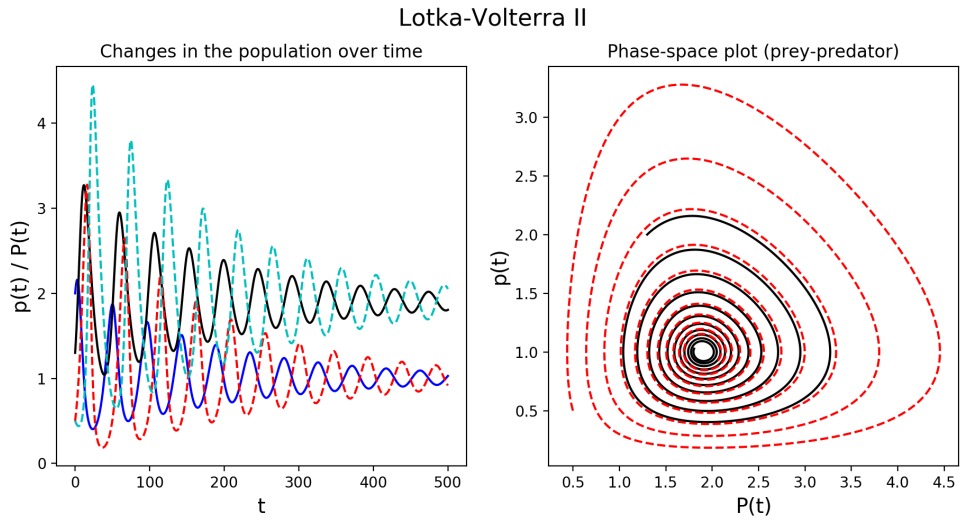

Figure 3: Left: changes in the prey population (blue line or dashed red line) and the predator population (black line or cyan dashed line), when initiated with $p(0) = 2.0$ and $P(0) = 1.3$ or $p(0) = 0.5$ and $P(0) = 0.5$. Right: phase-space plot using the previous two different initial conditions, but the same parameters (table 2) for the populations.

Evaluating figure 3 we can conclude, that running with the same parameters, but starting from different initial conditions will go towards the same equilibrium. So it does not depend on the starting populations, only on the parameters shown in table 2. It meets our expectations, that we consider, when interpreting (10) and (11).

The equilibrium state can be also calculated here for given parameters (see table 2) and we can get the figure we expect:

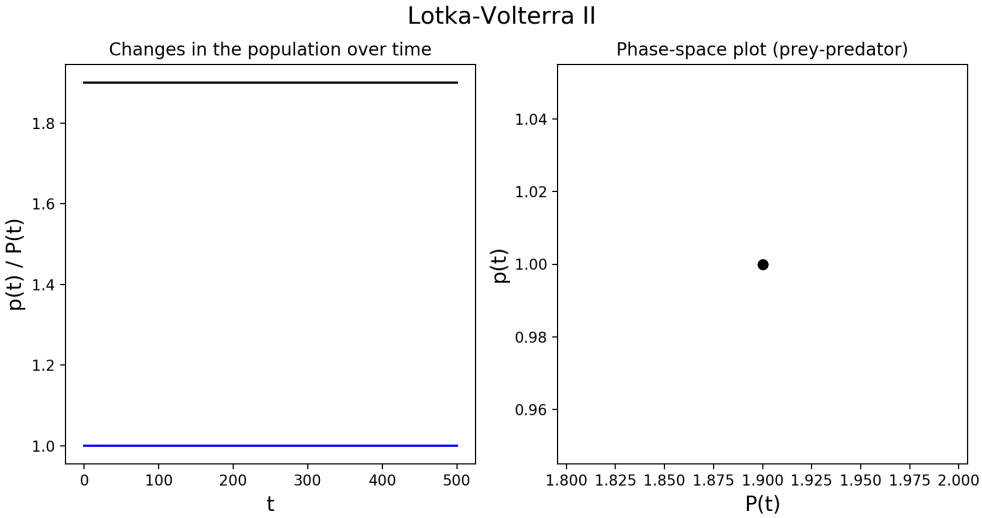

Figure 4: Equilibrium state, prey (blue), predator (black).

2.3 LVM-III

2.3.1 Background

We saw in LVM-II, that we get a much better model introducing the prey carrying capacity K, but still we are not satisfied. Further modifications are needed. Let us consider, that a predator needs time for searching the prey (t_{search}) and eating it($t_{handling}$), because it does not happen immediately. It is also wise to set a predator carrying capacity proportional to $p(t)$ as: $kp(t)$. After these modifications the two equations describing the system:

$$\frac{dp(t)}{dt} = ap(t)\left[1 - \frac{p(t)}{K}\right] - \frac{bp(t)P(t)}{1 + bp(t)t_h}, \text{ (LVM-III for prey)}, \tag{12}$$

where t_h is the handling time.

$$\frac{dP(t)}{dt} = mP(t)\left[1 - \frac{P(t)}{kp(t)}\right], \text{ (LVM-III for predator)}. \tag{13}$$

2.3.2 Results

To validate the obtained results, the same initial values and parameters were used as written in [1] and the t_h is set to hundred times the time step of the Runge-Kutta solver. (see Table 3).

Model	a	b	m	K	k	t_h
LVM-III	0.2	0.1	0.1	500	0.2	0.5

Table 3: Parameters used in the simulation.

We can observe the existence of three dynamic regimes as a function of the interaction rate (b) between predator and prey:

- small b: no oscillations, no overdamping,
- medium b: damped oscillations, which are converging to a stable equilibrium,
- large b: limit cycle.

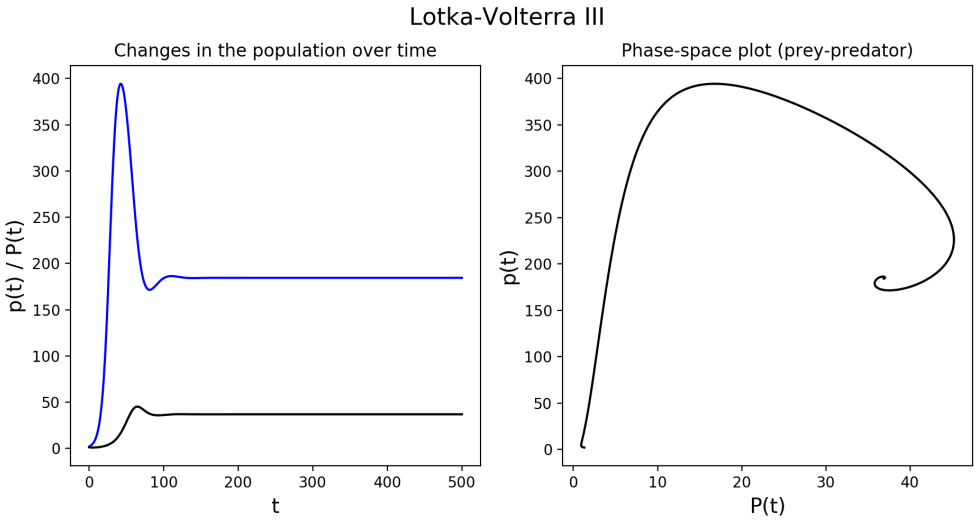

Figure 5: Case: $b = 0.005$ (small). Left: changes in the prey population (blue line) and the predator population (black line), when initiated with $p(0) = 2.0$ and $P(0) = 1.3$. Right: phase-space plot using the previous initial conditions.

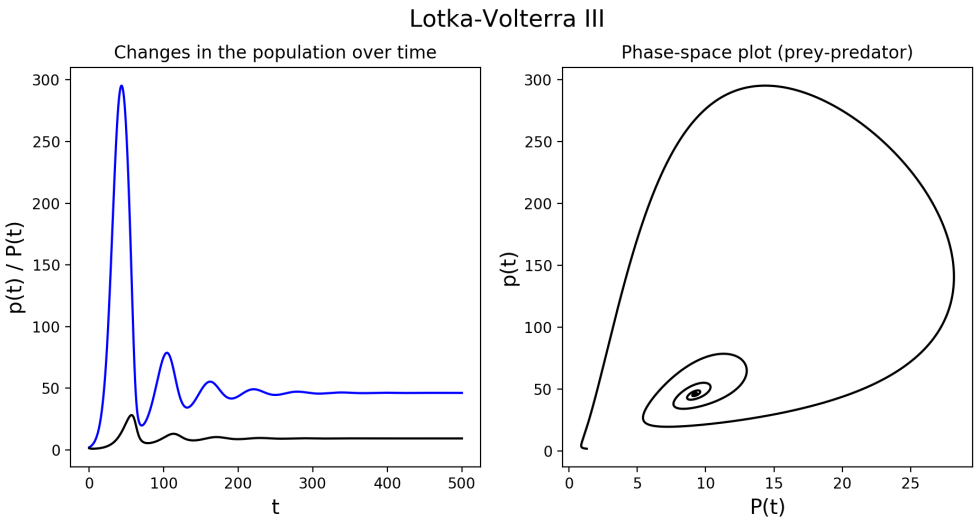

Figure 6: Case: $b = 0.036$ (medium). Left: changes in the prey population (blue line) and the predator population (black line), when initiated with $p(0) = 2.0$ and $P(0) = 1.3$. Right: phase-space plot using the previous initial conditions.

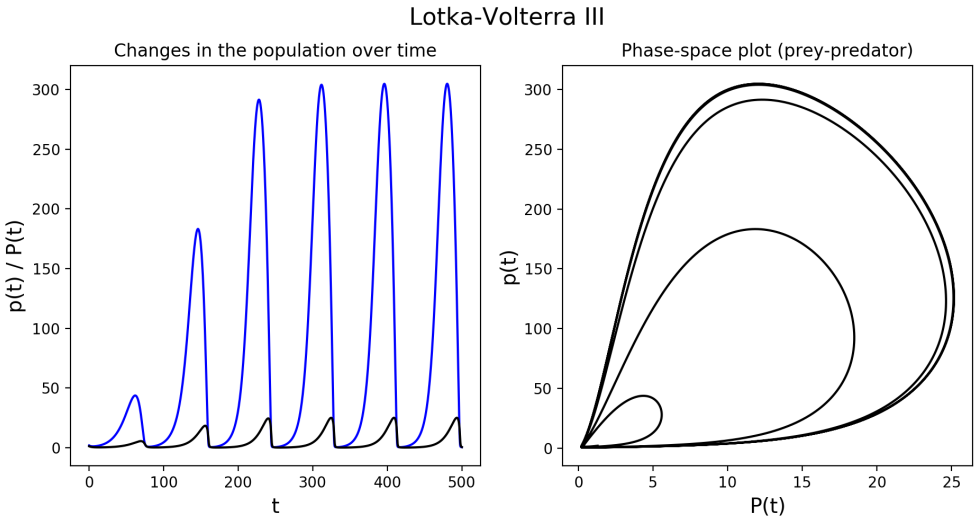

Figure 7: Case: $b = 5.0$ (large). Left: changes in the prey population (blue line) and the predator population (black line), when initiated with $p(0) = 2.0$ and $P(0) = 1.3$. Right: phase-space plot using the previous initial conditions.

The transition can also be seen at a specific parameter ($b = 0.219$) with the same initial conditions and parameters as before (except b, which is changing).

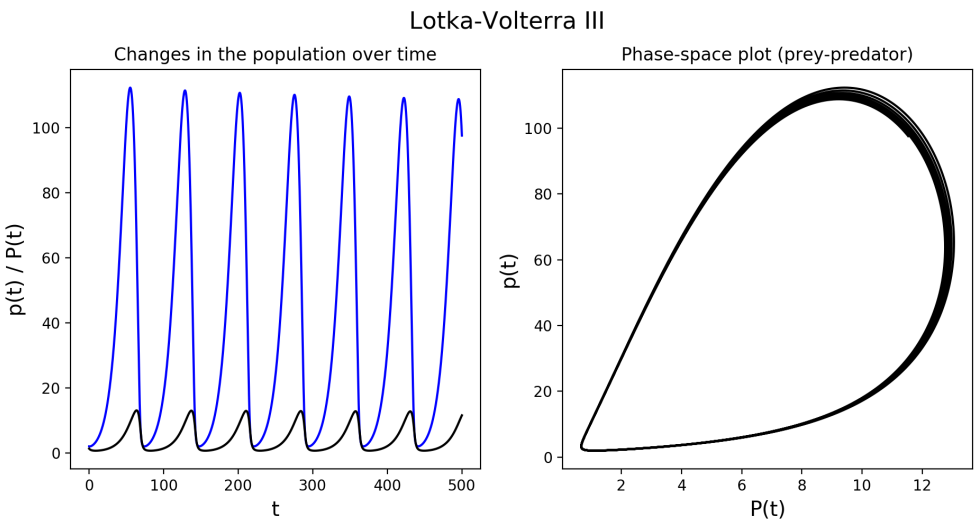

Figure 8: Case: $b = 0.0218$. Before transition.

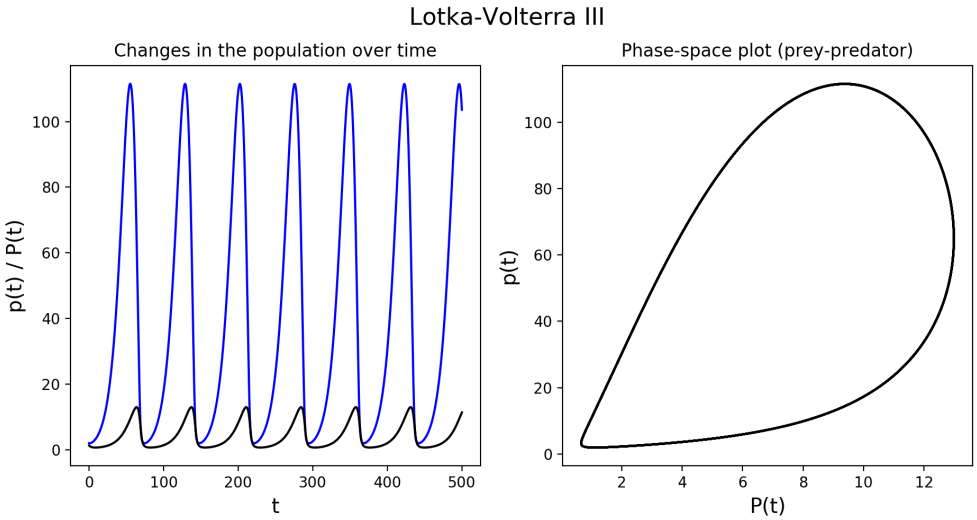

Figure 9: Case: $b = 0.0219$. At the transition.

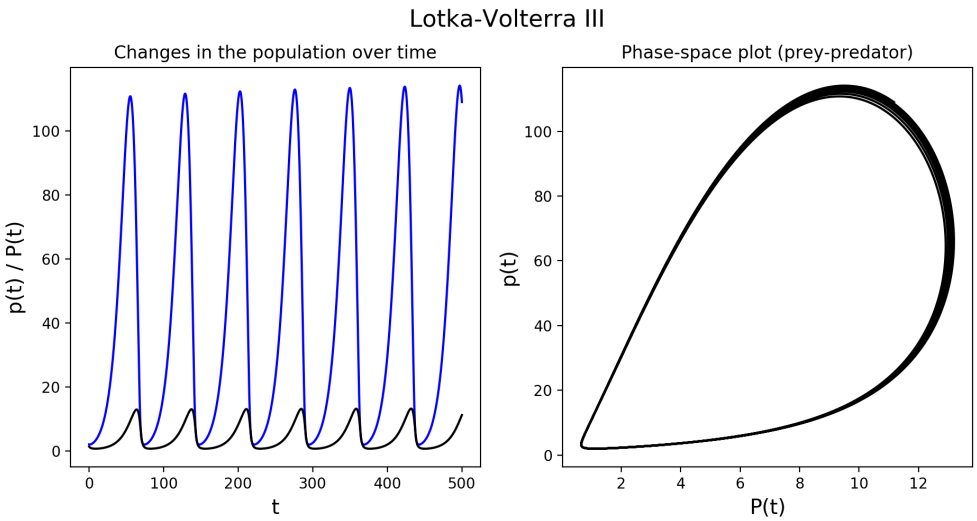

Figure 10: Case: $b = 0.0220$. After the transition.

We can conclude here, that this model is providing satisfactory results: the prey population can be kept from getting too large and from fluctuating widely and the changes in the parameters can lead to fluctuations or to nearly vanishing predators.

2.4 LVM two predators one prey

2.4.1 Background

As an additional examination it is interesting, what happens, when two predators share the same prey. The equations using LVM-II:

$$\frac{dp(t)}{dt} = ap(t)\left[1 - \frac{p(t)}{K}\right] - \left[b_1 P_1(t) + b_2 P_2(t)\right] p(t), \text{ (LVM-2pred for prey)}, \tag{14}$$

$$\frac{dP_1(t)}{dt} = \varepsilon_1 b_1 P_1(t) p(t) - m_1 P_1(t), \text{ (LVM-2pred for first predator)}, \tag{15}$$

$$\frac{dP_2(t)}{dt} = \varepsilon_2 b_2 P_2(t) p(t) - m_2 P_2(t), \text{ (LVM-2pred for second predator)}. \tag{16}$$

where the same notations are used to all parameters and the indices are introduced to let us set different skills for each predator.

2.4.2 Results

The parameter tables:

Model	a	b_1	ε_1	m_1	K
LVM-2pred	0.2	0.1	1.0	0.1	1.7

Table 4: Parameters used in the simulation for the first predator.

Model	a	b_1	ε_1	m_1	K
LVM-2pred	0.2	0.2	2.0	0.1	1.7

Table 5: Parameters used in the simulation for the second predator.

So the second predator is twice as effective as the first one in hunting and converting the prey into food.

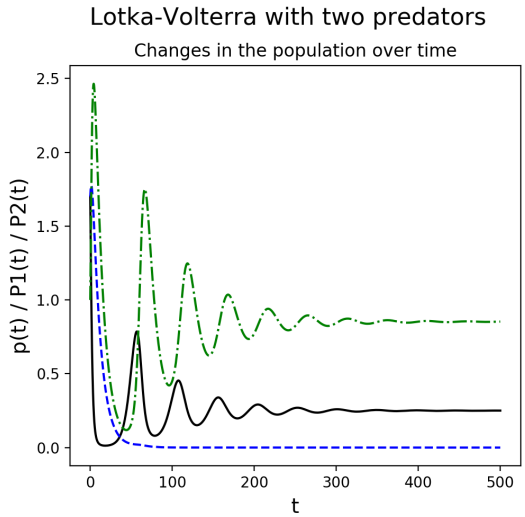

Figure 11: Changes in the prey population (black line), the first predator population (blue dashed line) and the second predator population (green dashed line), when initiated with $p(0) = 1.7$ and $P_1(0) = 1.7$ and $P_2(0) = 1.0$.

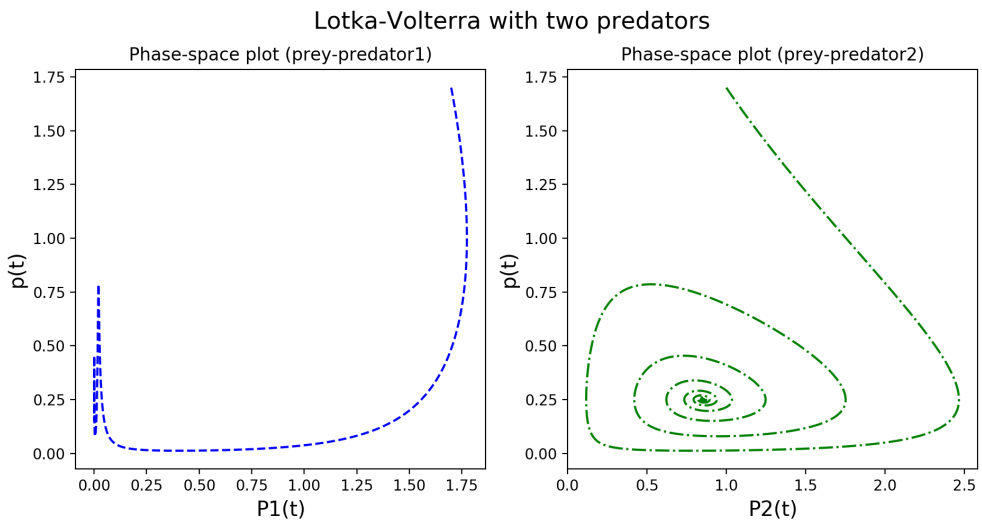

Figure 12: Phase-space plot. On the left with the first predator, on the right with the second one initiated with $p(0) = 1.7$ and $P_1(0) = 1.7$ and $P_2(0) = 1.0$.

Our assumption is proved. We can see, that the less-skilled predator die out, while the other one's population gets in equilibrium with the prey population. The results of an other run shows, that both predator can coexist if they have same hunting skills ($b_1 = b_2$ and $\varepsilon_1 = \varepsilon_2$).

3 Application in plasma physics

In this section I would like to briefly introduce an application in fusion plasma physics. There exists a theory to describe the behaviour of radially localized, sheared turbulent flows, which modifies the particle transport from inside the Tokamak device to the edges. This can cause - according to the theory - the L-H-transitions.[1]

If we consider, that many small-scale turbulent flows are created in a two-dimensional plane (formed by magnetic effects), then due to the inverse energy-cascade these small-scale structures can sustain a larger-scale sheared zonal flow. So those turbulent structures are responsible for the existence of this zonal flow.

Here we can make an identification of these structures: the small-scale turbulent flows are playing the role of the prey population, while the larger-scale sheared zonal flow is the predator, that uses the latter one as a supply.[2]

If we set the power supply (external heat, current) in a Tokamak a way, that it is close to the transition threshold, we can measure phase transitions from L-mode to H-mode. In between there are the Limit-Cycle-Oscillations (LCO). (see figure 13) We could also look for similar properties, compare this and what we saw in LVM-III varying b.

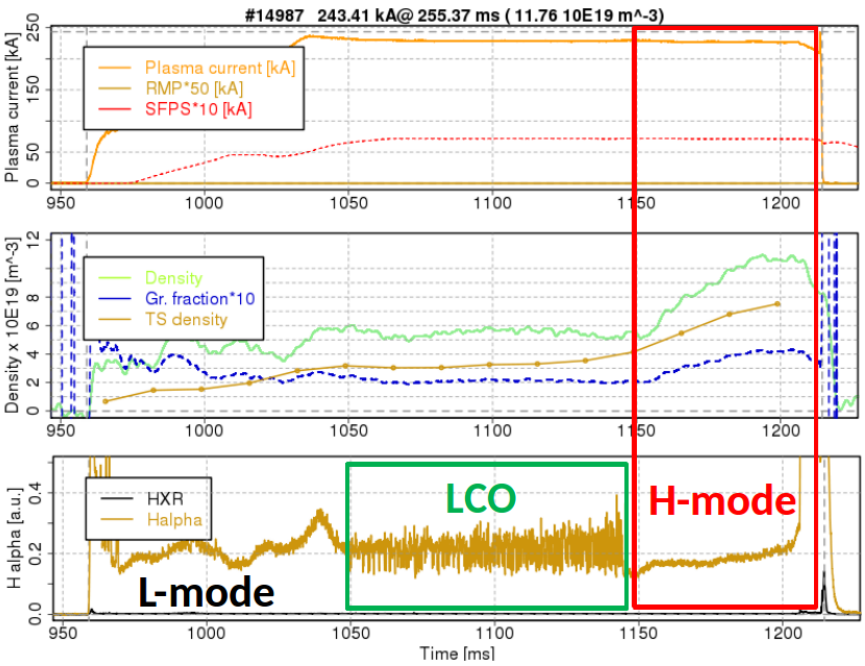

Figure 13: Measured profiles at Compass, Prague, SUMTRAIC2017, Task4.

[1]The plasma in a Tokamak device can be in low-confinement L or in high-confinement H mode. This means, that in those modes the plasma is less or more confined.

[2]This approach is rather only a brief demonstration. For more details read [2].

4 Conclusion

The numerical simulations with the post-processing are done in python and the results are shared with this work. One can see, that we got the expected results.

References

[1] Rubin H Landau, Manuel J Paez, Cristian Bordeianu: A Survey of Computational Physics - Introductory Computational Science, Princeton University Press, 2008.

[2] P H Diamond, S-I Itoh, K Itoh and T S Hahm: Zonal flows in plasma — a review, Plasma Phys. Control. Fusion **47** (2005) R35–R161

[3] (SUMTRAIC2017, Compass, Prague:

`http://www.ipp.cas.cz/strategie-av-21/Aktivity_2017/`)